Part II

Projective K-theory

Throughout this part we use the following notation: T is a group, 1 is its neutral element, K is the complex Hilbert space $l^2(T)$, $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subgroups of T the union of which is T, $T_0 := \{1\}$, E is a unital commutative C*-algebra, and f is a Schur E-function for T (Definition 5.0.1).

In the usual K-theory the orthogonal projections (used for K_0) and the unitaries (used for K_1) are identified with elements of the square matrices, which is not a very elegant procedure from the mathematical point of view, but is justified as a very efficient pragmatic solution. It seems to us that in the present more complicated construction the danger of confusion produced by these identifications is greater and we decided to separate these three domains. Unfortunately this separation complicates the presentation and the notation. Moreover, we also do identifications! In general the stability does not hold. We present in Theorem 6.3.3 (as an example) some strong conditions under which stability holds for K_0.

For projective representations of groups we use [2] (but the groups will be finite here) and for the K-theory we use [4], the construction of which we follow step by step. In the sequel we give a list of notation used in this Part.

1) We put for every involutive algebra F,

$$Pr F := \left\{ P \in F \mid P = P^* = P^2 \right\}$$

and for every $A \subset F$,

$$A^c := \{ x \in F \mid y \in A \Longrightarrow xy = yx \} .$$

2) We denote for every unital involutive algebra F by 1_F its unit and set

$$Un\, F := \{ U \in F \mid UU^* = U^*U = 1_F \} .$$

3) If F is a unital C*-algebra and $U, V \in Un\, F$ then we denote by $U \sim_h V$ the assertion U and V are homotopic in $Un\, F$ and put

$$Un_0\, F := \{ U \in Un\, F \mid U \sim_h 1_F \} .$$

Moreover $GL(F)$ denotes the group of invertible elements of F and $GL_0(F)$ the elements of $GL(F)$ which are homotopic to 1_F in $GL(F)$.

4) If F is a unital C*-algebra and G is a unital C*-subalgebra of F then we denote by $Un_G F$ the set of elements of $Un\, F$ which are homotopic to an element of $Un\, G$ in $Un\, F$ and by $GL_G(F)$ the set of elements of $GL(F)$ which are homotopic to an element of $GL(G)$ in $GL(F)$.

5) If Ω is a topological space, F a C*-algebra, and $A \subset F$ then we put

$$\mathscr{C}(\Omega,A) := \{\, X \in \mathscr{C}(\Omega,F) \mid \omega \in \Omega \Longrightarrow X(\omega) \in A \,\} \;.$$

6) Hilbert E-C*-algebra ([1] Definition 5.6.1.4).

7) $\mathscr{L}_E(H)$ ([1] Definition 5.6.1.7).

Chapter 5

Some Notation and the Axiom

DEFINITION 5.0.1 Let S be a group and let 1 be its neutral element. A **Schur E-function for S** is a map

$$f : S \times S \longrightarrow Un\, E$$

such that $f(1,1) = 1_E$ and

$$f(r,s)f(rs,t) = f(r,st)f(s,t)$$

for all $r,s,t \in T$. We denote by $\mathscr{F}(S,E)$ the set of Schur E-functions for S.

Schur functions are also called normalized factor set or multiplier or two-co-cycle (for S with values in $Un\, E$) in the literature.

DEFINITION 5.0.2 Let F be an full E-C*-algebra and $n \in \mathbb{N}^*$. We put for every $t \in T_n$, $\xi \in F^{T_n} = F \otimes l^2(T_n)$, and $x \in F$,

$$V_t\xi := V_t^F \xi : T_n \longrightarrow F\,, \quad s \longmapsto f(t, t^{-1}s)\xi(t^{-1}s)\,,$$

$$x \otimes id_K : F^{T_n} \longrightarrow F^{T_n}\,, \quad \xi \longmapsto (x\xi_s)_{s \in T_n}\,,$$

so we have

$$(x \otimes id_K)V_t\xi : T_n \longrightarrow F\,, \quad s \longmapsto f(t, t^{-1}s)x\xi(t^{-1}s)\,.$$

We define

$$F_n := \left\{ \sum_{t \in T_n} (X_t \otimes id_K)V_t \;\middle|\; (X_t)_{t \in T_n} \in F^{T_n} \right\}\,.$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{C}_E$ then we put

$$\varphi_n : F_n \longrightarrow G_n\,, \quad X \longmapsto \sum_{t \in T_n} ((\varphi X_t) \otimes id_{K_n})V_t\,.$$

F_n is a full E-C*-subalgebra of $\mathscr{L}_F(F^{T_n})$ (Proposition 4.1.7 b), [2] Theorem 2.1.9 h), k)), so $1_{F_n} = 1_E$, and φ_n is an E-C*-homomorphism, injective or surjective if φ is so ([2] Corollary 2.2.5). Moreover F_m is canonically a full E-C*-subalgebra of F_n for every $m \in \mathbb{N}^*$, $m < n$ ([2] Proposition 2.1.2). For every $n \in \mathbb{N}$, $F_n \times G_n \approx (F \times G)_n$.

DEFINITION 5.0.3 We fix in Part II a sequence $(C_n)_{n \in \mathbb{N}} \in \prod\limits_{n \in \mathbb{N}} E_n$, put

$$A_n := C_n^* C_n\,, \qquad B_n := C_n C_n^*\,,$$

and assume $A_n, B_n \in Pr\, E_n$, $A_n + B_n = 1_E = 1_{E_n}$, and $C_n \in (E_{n-1})^c$ for every $n \in \mathbb{N}$ (where we used the inclusion $E_{n-1} \subset E_n$ in the last relation).

From

$$A_n = A_n(A_n + B_n) = A_n^2 + A_n B_n = A_n + A_n B_n\,,$$

$$C_n = C_n(A_n + B_n) = C_n A_n + C_n B_n = C_n + C_n^2 C_n^*$$

we get $A_n B_n = C_n^2 = 0$ for every $n \in \mathbb{N}$.

We have $C_n \in (F_{n-1})^c$ for every $n \in \mathbb{N}$ and for every full E-C*-algebra F (where we used the inclusion $F_{n-1} \subset F_n$).

DEFINITION 5.0.4 Let $(S_m)_{m\in\mathbb{N}}$ be a sequence of finite groups and $(k_n)_{n\in\mathbb{N}}$ a strictly increasing sequence in $\mathbb{N}$ such that $T_n = \prod_{m=1}^{k_n} S_m$ for all $n \in \mathbb{N}$. We identify S_m with a subgroup of T for every $m \in \mathbb{N}$. Assume that for every $m \in \mathbb{N}$ there is a $g_m \in \mathscr{F}(S_m, E)$ such that

$$f(s,t) = \prod_{m\in\mathbb{N}} g_m(s_m, t_m)$$

for all $s,t \in T$. For every $n \in \mathbb{N}$ let $m \in \mathbb{N}$, $k_{n-1} < m \le k_n$, let $\chi : \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow S_m$ be an injective group homomorphism, and $\beta_1, \beta_2 \in Un\, E$. We put

$$a := \chi(1,0)\,, \qquad b := \chi(0,1)\,, \qquad \alpha_1 := f(a,a)\,, \qquad \alpha_2 := f(b,b)\,,$$

$$C_n := \frac{1}{2}((\beta_1 \otimes id_K)V_a^f + (\beta_2 \otimes id_K)V_b^f)\,.$$

If $f(a,b) = -f(b,a) = 1_E$ and $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$ then $(C_n)_{n\in\mathbb{N}}$ fulfills the conditions of *Axiom 5.0.3*.

The assertion follows from [2] Theorem 2.2.18 a), b). ■

Remark 1. If $E = \mathbb{C}$, $S_m = \mathbf{Z}_2 \times \mathbf{Z}_2$, and $k_m = m$ for every $m \in \mathbb{N}$ then (by [2] Proposition 3.2.1 c) and [2] Corollary 3.2.2 d)) we may choose $(C_n)_{n\in\mathbb{N}}$ in such a way that the corresponding K-theory coincides with the classical one.

Remark 2. Denote by T_n the set of permutations p of $\mathbb{N}$ such that $\{\, j \in \mathbb{N} \mid p(j) \neq j \,\} \subset \mathbb{N}_{4n}$ so T is the set of permutations p of $\mathbb{N}$ such that $\{\, j \in \mathbb{N} \mid p(j) \neq j \,\}$ is finite. This example shows that the given conditions for T_n in Example 5.0.4 are not automatically fulfilled.

Chapter 6

The Functor K_0

6.1 K_0 for $\mathfrak{C}_E$

Throughout this section F denotes a full E-C*-algebra.

PROPOSITION 6.1.1 *Let $n \in \mathbb{N}$.*

a) $A_n, B_n \in (F_{n-1})^c$ (where we used the inclusion $F_{n-1} \subset F_n$).

b) $A_n F_n A_n$ is a unital C-algebra with A_n as unit.*

c) The map

$$\bar{\rho}_n^F : F_{n-1} \longrightarrow F_n\,, \quad X \longmapsto A_n X = X A_n = A_n X A_n = C_n^* X C_n$$

(where we used the inclusion $F_{n-1} \subset F_n$) is an E-linear injective C-homomorphism.*

Only the injectivity of $\bar{\rho}_n^F$ needs a proof. Let $X \in F_{n-1}$ with $\bar{\rho}_n^F X = 0$. Then

$$C_n^* C_n X = 0\,, \qquad X C_n = C_n X = 0\,,$$

$$X B_n = X C_n C_n^* = 0\,, \qquad X = X(A_n + B_n) = 0\,. \qquad \blacksquare$$

Remark. $\bar{\rho}_n^F$ is not unital since $\bar{\rho}_n^F 1_E = A_n$.

DEFINITION 6.1.2 *We put for all $m, n \in \mathbb{N}$, $m < n$,*

$$\rho_{n,m}^F := \bar{\rho}_n^F \circ \bar{\rho}_{n-1}^F \circ \cdots \circ \bar{\rho}_{m+1}^F : F_m \longrightarrow F_n\,.$$

Then $\{(F_n)_{n\in\mathbb{N}}, (\rho_{n,m}^F)_{n,m\in\mathbb{N}}\}$ is an inductive system of full E-C-algebras with injective E-linear (but not unital) maps. We denote by $\{F_\rightarrow, (\rho_n^F)_{n\in\mathbb{N}}\}$ its algebraic inductive limit. $F_\rightarrow$ is an involutive (but not unital) algebra endowed with the structure of an algebraic E-C*-algebra, ρ_n^F is injective and E-linear for every $n \in \mathbb{N}$, and $(Im\,\rho_n^F)_{n\in\mathbb{N}}$ is an increasing sequence of involutive subalgebras and algebraic E-C*-subalgebras of $F_\rightarrow$ the union of which is $F_\rightarrow$. We put for every $X \in F_n$,*

$$X_\rightarrow := X_{\rightarrow n} := X_{\rightarrow n}^F := \rho_n^F X\,,$$

and

$$1_{\to n} := 1^F_{\to n} := \rho^F_n 1_{F_n} = \rho^F_n 1_E\,,$$

$$F_{\to n} := Im\,\rho^F_n\ .$$

In particular

$$(A_n)_{\to} = \rho^F_n A_n = 1_{\to, n-1}, \qquad (B_n)_{\to} = \rho^F_n B_n, \qquad (C_n)_{\to} = \rho^F_n C_n\ .$$

We put

$$Pr\,F_{\to} := \left\{\, P \in F_{\to} \mid P = P^* = P^2 \,\right\} = \bigcup_{n\in\mathbb{N}} (Pr\,F_{\to n})\ .$$

For $P,Q \in Pr\,F_{\to}$ we put $P \sim_0 Q$ if there is an $X \in F_{\to}$ with $X^*X = P$, $XX^* = Q$ (in this case there is an $n \in \mathbb{N}$ such that $P,Q,X \in F_{\to n}$); $\sim_0$ is the Murray - von Neumann equivalence relation, which we shall use also in the case of C-algebras. For every $P \in Pr\,F_{\to}$ we denote by $\dot{P}$ its equivalence class in $Pr\,F/\sim_0$.*

Often we shall identify F_n with $F_{\to n}$ by using ρ^F_n. By this identification $F_{\to n}$ is a full E-C*-algebra with $1_{\to n}$ as unit.

$F_{\to}$ is also endowed with a C*-norm and its completion in this norm is the C*-inductive limit of the above inductive system, but we shall not use this supplementary structure in the sequel.

PROPOSITION 6.1.3 *If $n \in \mathbb{N}$ and $P \in Pr\,F_{\to, n-1}$ then*

$$P = (A_n)_{\to} P \sim_0 (B_n)_{\to} P = (C_n)_{\to} P (C_n)^*_{\to}\ .$$

We have

$$((C_n)_{\to}P)^*((C_n)_{\to}P) = P(C_n)^*_{\to}(C_n)_{\to}P = (A_n)_{\to}P\,,$$

$$((C_n)_{\to}P)((C_n)_{\to}P)^* = P(C_n)_{\to}(C_n)^*_{\to}P = (B_n)_{\to}P\,,$$

so $(A_n)_{\to}P \sim_0 (B_n)_{\to}P$. ■

PROPOSITION 6.1.4 *For every finite family $(P_i)_{i\in I}$ in $Pr\,F_{\to}$ there is a family $(Q_i)_{i\in I}$ in $Pr\,F_{\to}$ such that $P_i \sim_0 Q_i$ for every $i \in I$ and $Q_iQ_j = 0$ for all distinct $i, j \in I$.*

We prove the assertion by complete induction with respect to $Card\, I$. Let $i_0 \in I$ and put $J := I \setminus \{i_0\}$. We may assume, by the induction hypothesis, that there is an $n \in \mathbb{N}$ with $P_i \in Pr\, F_{\rightarrow,n-1}$ for all $i \in I$ and $P_i P_j = 0$ for all distinct $i, j \in J$. By Proposition 6.1.3,

$$P_{i_0} = (A_n)_{\rightarrow} P_{i_0} \sim_0 (C_n)_{\rightarrow} P_{i_0} (C_n)^*_{\rightarrow} =: Q_{i_0}\,,$$

and

$$Q_{i_0} P_j = (C_n)_{\rightarrow} P_{i_0} (C_n)^*_{\rightarrow} (A_n)_{\rightarrow} P_j = (C_n)_{\rightarrow} P_{i_0} (C_n^* A_n)_{\rightarrow} P_j = 0$$

for all $j \in J$. ■

PROPOSITION 6.1.5 *Let $P, Q \in Pr\, F_{\rightarrow}$.*

a) If $P', P'', Q', Q'' \in Pr\, F_{\rightarrow}$ such that

$$P \sim_0 P' \sim_0 P'', \qquad Q \sim_0 Q' \sim_0 Q'', \qquad P'Q' = P''Q'' = 0$$

then

$$P' + Q' \sim_0 P'' + Q''\,.$$

We put

$$\dot{P} \oplus \dot{Q} := \overbrace{P' + Q'}^{\cdot}\,.$$

b) $Pr\, F_{\rightarrow} / \sim_0$ endowed with the above composition law $\oplus$ is an additive semi-group with $\dot{0}$ as neutral element. We denote by $K_0(F)$ its associated Grothendieck group and by

$$[\,\cdot\,]_0 : Pr\, F_{\rightarrow} \longrightarrow K_0(F)$$

the Grothendieck map ([4] 3.1.1).

c) $K_0(F) = \{\, [P]_0 - [Q]_0 \mid P, Q \in Pr\, F_{\rightarrow} \}$.

d) For every $a \in K_0(F)$ there are $P, Q \in Pr\, F_{\rightarrow}$ and $n \in \mathbb{N}$ such that

$$P = P(A_n)_{\rightarrow}, \qquad Q = Q(B_n)_{\rightarrow}, \qquad a = [P]_0 - [Q]_0\,.$$

a) Let $X, Y \in F_{\rightarrow}$ with

$$X^*X = P', \qquad XX^* = P'', \qquad Y^*Y = Q', \qquad YY^* = Q''\,.$$

Then

$$0 = P'Q' = X^*XY^*Y, \qquad 0 = P''Q'' = XX^*YY^*$$

so

$$XY^* = X^*Y = 0, \qquad (X+Y)^*(X+Y) = X^*X + Y^*Y = P' + Q',$$

$$(X+Y)(X+Y)^* = XX^* + YY^* = P'' + Q'', \qquad P' + Q' \sim_0 P'' + Q''.$$

b) and c) follow from a) and Proposition 6.1.4.

d) follows from c) and Proposition 6.1.3. ■

COROLLARY 6.1.6 *The following are equivalent for all $n \in \mathbb{N}$ and $P, Q \in Pr\, F_{\to n}$.*

a) $[P]_0 = [Q]_0$.

b) There is an $R \in Pr\, F_{\to}$ such that

$$PR = QR = 0, \qquad P + R \sim_0 Q + R.$$

c) There is an $m \in \mathbb{N}$, $m > n+1$, such that

$$P + (B_m)_{\to} \sim_0 Q + (B_m)_{\to}$$

or (by identifying F_m with $F_{\to m}$)

$$\left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right) \sim_0 \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right).$$

$a \Rightarrow b$ follows from Proposition 6.1.4 (and from the definition of the Grothendieck group).

$b \Rightarrow c$. We may assume $R \in F_{\to, m-1}$ for some $m > n+1$. By Proposition 6.1.3,

$$P + (B_m)_{\to}R \sim_0 P + R \sim_0 Q + R \sim_0 Q + (B_m)_{\to}R,$$

so

$$P + (B_m)_{\to} = P + (B_m)_{\to}R + ((B_m)_{\to} - (B_m)_{\to}R) \sim_0$$

$$\sim_0 Q + (B_m)_{\to}R + ((B_m)_{\to} - (B_m)_{\to}R) = Q + (B_m)_{\to}.$$

It follows

$$\left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right) = \rho^F_{m,n} P + B_m + \left(A_m - \prod_{i=n+1}^{m} A_i\right) \sim_0$$

$$\sim_0 \rho^F_{m,n} Q + B_m + \left(A_m - \prod_{i=n+1}^{m} A_i\right) = \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right) .$$

c⇒a is trivial. ■

COROLLARY 6.1.7 *If for every $n \in \mathbb{N}$ and $P \in Pr\, F_{\to n}$ there is an $m \in \mathbb{N}$, $m > n+1$, such that $P + (B_m)_{\to} \sim_0 1_E$ then $K_0(F) = \{0\}$.*

Let $P, Q \in Pr\, F_{\to}$. By our hypothesis there is an $m \in \mathbb{N}$ such that $P + (B_m)_{\to} \sim_0 Q + (B_m)_{\to}$. By Corollary 6.1.6 $c \Rightarrow a$, $[P]_0 = [Q]_0$. Thus by Proposition 6.1.5 c), $K_0(F) = \{0\}$.
■

COROLLARY 6.1.8 $K_0(E) \neq \{0\}$.

Assume $K_0(E) = \{0\}$. Then $[1_E]_0 = [0]_0$, so by Corollary 6.1.6 $a \Rightarrow c$, there is an $n \in \mathbb{N}$ such that

$$1_E \sim_0 1_E - \prod_{i=1}^{n} A_i .$$

Let ω be a point of the spectrum of E. Since $E_n(\omega)$ is a product of square matrices the above relation leads to a contradiction by using the trace function. ■

PROPOSITION 6.1.9 *Let $\mathscr{G}$ be an additive group and $\nu : Pr\, F_{\to} \to \mathscr{G}$ a map such that*

1) $P, Q \in Pr\, F_{\to}, PQ = 0 \implies \nu(P+Q) = \nu(P) + \nu(Q)$.

2) $P, Q \in Pr\, F_{\to}, P \sim_0 Q \implies \nu(P) = \nu(Q)$.

Then there is a unique group homomorphism $\mu : K_0(F) \to \mathscr{G}$ such that $\mu[P]_0 = \nu(P)$ for every $P \in Pr\, F_{\to}$.

By 2), ν is well-defined on $Pr F_{\rightarrow}/\sim_0$ and by 1) and Proposition 6.1.5 a),b), ν is an additive map on $Pr F_{\rightarrow}/\sim_0$. By 2) and Corollary 6.1.6 a$\Rightarrow$b, ν is well-defined on $K_0(F)$. The existence and uniqueness of μ with the given properties follows now from Proposition 6.1.5 c). ■

PROPOSITION 6.1.10 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{C}_E$.*

a) For $m,n \in \mathbb{N}$, $m < n$, the diagram

$$\begin{array}{ccc} F_m & \xrightarrow{\rho^F_{n,m}} & F_n \\ {\scriptstyle\varphi_m}\downarrow & & \downarrow{\scriptstyle\varphi_n} \\ G_m & \xrightarrow[\rho^G_{n,m}]{} & G_n \end{array}$$

is commutative. Thus there is a unique E-linear involutive algebra homomorphism $\varphi_{\rightarrow} : F_{\rightarrow} \longrightarrow G_{\rightarrow}$ with

$$\varphi_{\rightarrow} \circ \rho^F_n = \rho^G_n \circ \varphi_n$$

for every $n \in \mathbb{N}$.

b) $\varphi_{\rightarrow}$ is injective or surjective if φ is so.

c) There is a unique group homomorphism $K_0(\varphi) : K_0(F) \longrightarrow K_0(G)$ such that

$$K_0(\varphi)[P]_0 = [\varphi_{\rightarrow} P]_0$$

for every $P \in Pr F_{\rightarrow}$.

d) If φ is the identity map then $K_0(\varphi)$ is also the identity map.

e) If $\varphi = 0$ then $K_0(\varphi) = 0$.

a) It is sufficient to prove the assertion for $n = m+1$. For $X \in F_m$,

$$\varphi_n \bar{\rho}^F_n X = \varphi_n(A_n X) = A_n \varphi_n X = \bar{\rho}^G_n \varphi_n X$$

(where we used the inclusion $F_m \subset F_n$).

b) follows from the fact that for every $n \in \mathbb{N}$, φ_n is injective or surjective if φ is so ([2] Theorem 2.1.9 a))).

c) By a) and Proposition 6.1.3, the map

$$Pr\,F_{\rightarrow} \longrightarrow K_0(G), \quad P \longmapsto [\varphi_{\rightarrow}P]_0$$

possesses the properties from Proposition 6.1.9.

d) and e) are obvious. ■

COROLLARY 6.1.11 *If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ are morphisms in $\mathfrak{C}_E$ then*

$$(\psi \circ \varphi)_{\rightarrow} = \psi_{\rightarrow} \circ \varphi_{\rightarrow}, \qquad K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)\,.$$ ■

PROPOSITION 6.1.12

a) The maps

$$\mu : \check{F} \longrightarrow F\,, \quad (\alpha, x) \longmapsto \alpha + x\,,$$

$$\lambda' : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha, -\alpha)$$

are E-C-homomorphisms.*

b)

$$\mu \circ \iota^F = id_F\,, \qquad \iota^F \circ \mu + \lambda' \circ \pi^F = id_{\check{F}}\,,$$

$$K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F) = id_{K_0(\check{F})}\,.$$

c)

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota^F)} K_0(\check{F}) \underset{K_0(\lambda^F)}{\overset{K_0(\pi^F)}{\rightleftarrows}} K_0(E) \longrightarrow 0$$

is a split exact sequence.

a) is easy to see.

b) For $(\alpha, x), (\beta, y) \in \check{F}$,

$$\iota^F \mu(\alpha, x) = (0, \alpha + x), \qquad \lambda' \pi^F(\alpha, x) = (\alpha, -\alpha)\,,$$

$$(\iota^F \mu(\alpha, x))(\lambda' \pi^F(\beta, y)) = (0, \alpha + x)(\beta, -\beta) = (0, 0)\,,$$

$$(\iota^F \mu + \lambda' \pi^F)(\alpha, x) = (\alpha, x)$$

so $\iota^F \circ \mu + \lambda' \circ \pi^F$ is a full E-C*-homomorphism and

$$\iota^F \circ \mu + \lambda' \circ \pi^F = id_{\check{F}} \ .$$

By a) and Corollary 6.1.11,

$$\iota^F_{\rightarrow} \circ \mu_{\rightarrow} + \lambda'_{\rightarrow} \circ \pi^F_{\rightarrow} = id_{\check{F}_{\rightarrow}} \ .$$

By Proposition 6.1.10 c),d) and Corollary 6.1.11, for $P \in Pr\,\check{F}_{\rightarrow}$,

$$(K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F))[P]_0 = K_0(\iota^F \circ \mu)[P]_0 + K_0(\lambda' \circ \pi^F)[P]_0 =$$

$$= [\iota^F_{\rightarrow} \mu_{\rightarrow} P]_0 + [\lambda'_{\rightarrow} \pi^F_{\rightarrow} P]_0 = [(\iota^F \circ \mu + \lambda' \circ \pi^F)_{\rightarrow} P]_0 = [P]_0$$

so by Proposition 6.1.5 c),

$$K_0(\iota^F) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi^F) = id_{K_0(\check{F})} \ .$$

c) By b), Proposition 6.1.10 d),e), and Corollary 6.1.11,

$$K_0(\pi^F) \circ K_0(\iota^F) = K_0(\pi^F \circ \iota^F) = 0 \,,$$

$$K_0(\pi^F) \circ K_0(\lambda^F) = K_0(\pi^F \circ \lambda^F) = id_{K_0(E)} \,,$$

$$K_0(\mu) \circ K_0(\iota^F) = K_0(\mu \circ \iota^F) = id_{K_0(F)}$$

and so $K_0(\iota^F)$ is injective. By b), for $a \in K_0(\check{F})$,

$$a = K_0(\iota^F) K_0(\mu) a + K_0(\lambda') K_0(\pi^F) a \ .$$

Thus if $a \in Ker\,K_0(\pi^F)$ then $a = K_0(\iota^F) K_0(\mu) a \in Im\,K_0(\iota^F)$, and so $Ker\,K_0(\pi^F) = Im\,K_0(\iota^F)$.

6.2 K_0 for $\mathfrak{M}_E$

DEFINITION 6.2.1 *Let F be an E-C*-algebra and consider the split exact sequence*

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\rightarrow}} E \longrightarrow 0$$

introduced in Definition 4.1.4. *We put*

$$K_0(F) := Ker\,K_0(\pi^F) \ .$$

By Proposition 6.1.12 c), this definition does not contradict the definition given in Proposition 6.1.5 b) for the case that F is an full E-C*-algebra .

$K_0(\{0\}) = \{0\}$ since $\pi^{\{0\}}$ is bijective.

PROPOSITION 6.2.2 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$.*

a) The diagram

$$\begin{array}{ccccc} F & \xrightarrow{\iota^F} & \check{F} & \xrightarrow{\pi^F} & E \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\check{\varphi}} & & \| \\ G & \xrightarrow[\iota^G]{} & \check{G} & \xrightarrow[\pi^G]{} & E \end{array}$$

is commutative.

b) The diagram

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{\subset} & K_0(\check{F}) & \xrightarrow{K_0(\pi^F)} & K_0(E) \\ {\scriptstyle K_0(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle K_0(\check{\varphi})} & & \| \\ K_0(G) & \xrightarrow[\subset]{} & K_0(\check{G}) & \xrightarrow[K_0(\pi^G)]{} & K_0(E) \end{array}$$

is commutative, where $K_0(\varphi)$ is defined by $K_0(\check{\varphi})$.

c) If $P \in Pr\, F_{\rightarrow}$ then

$$K_0(\varphi)[P]_0 = [\varphi_{\rightarrow} P]_0 \,.$$

d) $K_0(id_F) = id_{K_0(F)}$.

e) If $\varphi = 0$ then $K_0(\varphi) = 0$.

a) is obvious.

b) By a) and Corollary 6.1.11, the right part of the diagram is commutative. This implies the existence (and uniqueness) of $K_0(\varphi)$.

c) By a), b), Proposition 6.1.10 a),c), and Corollary 6.1.11,

$$K_0(\varphi)[P]_0 = K_0(\check{\varphi})[\iota^F_{\rightarrow} P]_0 = [\check{\varphi}_{\rightarrow} \iota^F_{\rightarrow} P]_0 = [\iota^G_{\rightarrow} \varphi_{\rightarrow} P]_0 = [\varphi_{\rightarrow} P]_0 \,.$$

d) and e) follow from c) and Proposition 6.1.5 c). ■

COROLLARY 6.2.3 *Let* $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ *be morphisms in* $\mathfrak{M}_E$.

a) $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$.

b) If φ *is an isomorphism then* $K_0(\varphi)$ *is also an isomorphism and*

$$K_0(\varphi)^{-1} = K_0(\varphi^{-1}) .$$

a) follows from Proposition 4.1.5 b), Corollary 6.1.11, and Proposition 6.2.2 b).

b) follows from a) and Proposition 6.2.2 d). ■

PROPOSITION 6.2.4 *For every E-C*-algebra F,*

$$K_0(F) = \left\{ [P]_0 - [\sigma^F_{\rightarrow} P]_0 \;\middle|\; P \in Pr\check{F}_{\rightarrow} \right\} .$$

For $P \in Pr\check{F}_{\rightarrow}$, by Proposition 6.2.2 c) and Corollary 6.1.11 (since $\pi^F = \pi^F \circ \sigma^F$),

$$K_0(\pi^F)[\sigma^F_{\rightarrow} P]_0 = [\pi^F_{\rightarrow} \sigma^F_{\rightarrow} P]_0 = [\pi^F_{\rightarrow} P]_0 = K_0(\pi^F)[P]_0$$

so

$$[P]_0 - [\sigma^F_{\rightarrow} P]_0 \in Ker\, K_0(\pi^F) = K_0(F) .$$

Let $a \in K_0(F)$. By Proposition 6.1.5 d), there are $Q, R \in Pr\check{F}_{\rightarrow}$ and $n \in \mathbb{N}$ such that

$$Q = Q(A_n)_{\rightarrow} , \qquad R = R(B_n)_{\rightarrow} , \qquad a = [Q]_0 - [R]_0 .$$

Then

$$a = [Q(A_n)_{\rightarrow}]_0 + [(B_n)_{\rightarrow} - R(B_n)_{\rightarrow}]_0 - ([R(B_n)_{\rightarrow}]_0 - [(B_n)_{\rightarrow} - R(B_n)_{\rightarrow}]_0) =$$

$$= [Q(A_n)_{\rightarrow} + ((B_n)_{\rightarrow} - R(B_n)_{\rightarrow})]_0 - [(B_n)_{\rightarrow}]_0 .$$

If we put

$$P := Q(A_n)_{\rightarrow} + ((B_n)_{\rightarrow} - R(B_n)_{\rightarrow})$$

then

$$a = [P]_0 - [(B_n)_{\rightarrow}]_0 .$$

By Proposition 6.2.2 c) and Corollary 6.1.11 (and Definition 4.1.4)

$$0 = K_0(\pi^F)a = K_0(\pi^F)[P]_0 - K_0(\pi^F)[(B_n)_\rightarrow]_0 = [\pi^F_\rightarrow P]_0 - [\pi^F_\rightarrow (B_n)_\rightarrow]_0 \,,$$

$$[\sigma^F_\rightarrow P]_0 = [\lambda^F_\rightarrow \pi^F_\rightarrow P]_0 = K_0(\lambda^F)[\pi^F_\rightarrow P]_0 = K_0(\lambda^F)[\pi^F_\rightarrow (B_n)_\rightarrow]_0 =$$

$$= [\lambda^F_\rightarrow \pi^F_\rightarrow (B_n)_\rightarrow]_0 = [\sigma^F_\rightarrow (B_n)_\rightarrow]_0 = [(B_n)_\rightarrow]_0 \,,$$

$$a = [P]_0 - [\sigma^F_\rightarrow P]_0 \,.$$

■

PROPOSITION 6.2.5 *Let F be an full E-C*-algebra and $n \in \mathbb{N}$.*

a) $C_n + C_n^* \in Un_0\, E_n$.

b) For $X, Y \in F_{n-1}$,

$$(C_n + C_n^*)(A_n X + B_n Y)(C_n + C_n^*) = B_n X + A_n Y \,.$$

c) If $U, V \in Un\, F_{n-1}$ *then* $A_n U + B_n V \in Un\, F_n$.

d) If $U \in Un\, F_{n-1}$ *then* $A_n U + B_n \in Un\, F_n$ *and* $A_n U + B_n U^* \in Un_0\, F_n$.

a) From

$$(C_n + C_n^*)(C_n + C_n^*) = B_n + A_n = 1_E$$

it follows that $C_n + C_n^*$ is unitary. Being selfadjoint, its spectrum is contained in $\{-1, +1\}$ and so it belongs to $Un_0\, E_n$ ([4] Lemma 2.1.3 (ii)).

b) We have

$$(C_n + C_n^*)(A_n X + B_n Y)(C_n + C_n^*) = (C_n X + C_n^* Y)(C_n + C_n^*) = B_n X + A_n Y \,.$$

c) We have

$$(A_n U + B_n V)(A_n U + B_n V)^* = A_n + B_n = 1_E \,,$$

$$(A_n U + B_n V)^*(A_n U + B_n V) = A_n + B_n = 1_E \,.$$

d) By c), $A_n U + B_n \in Un\, F_n$. By b),

$$(C_n + C_n^*)(A_n U^* + B_n)(C_n + C_n^*) = B_n U^* + A_n \,,$$

so it follows from a), that $A_nU^* + B_n$ is homotopic to $B_nU^* + A_n$ in $Un\, F_n$ and so

$$A_nU + B_nU^* = (A_nU + B_n)(A_n + B_nU^*)$$

is homotopic in $Un\, F_n$ to

$$(A_nU + B_n)(A_nU^* + B_n) = A_n + B_n = 1_E\,,$$

i.e. $A_nU + B_nU^* \in Un_0\, F_n$. ∎

PROPOSITION 6.2.6 *Let F be a full E-C*-algebra, $n \in \mathbb{N}$, $P, Q \in Pr\, F_n$, and $X \in F_n$ with $X^*X = P$, $XX^* = Q$. Then there is a $U \in Un_0\, F_{n+2}$ with*

$$U(A_{n+2}A_{n+1}P)U^* = A_{n+2}A_{n+1}Q\,, \qquad \text{i.e.} \qquad U_{\rightarrow}P_{\rightarrow}U^*_{\rightarrow} = Q_{\rightarrow}\,.$$

We have $X(1_E - P) = (1_E - Q)X = 0$. Put

$$V := A_{n+1}X + C_{n+1}(1_E - P) + C^*_{n+1}(1_E - Q) + B_{n+1}X^* \qquad (\in F_{n+1})\,.$$

Then

$$V^* = A_{n+1}X^* + C^*_{n+1}(1_E - P) + C_{n+1}(1_E - Q) + B_{n+1}X\,,$$

$$VV^* = A_{n+1}Q + B_{n+1}(1_E - P) + A_{n+1}(1_E - Q) + B_{n+1}P = A_{n+1} + B_{n+1} = 1_E\,,$$

$$V^*V = A_{n+1}P + A_{n+1}(1_E - P) + B_{n+1}(1_E - Q) + B_{n+1}Q = A_{n+1} + B_{n+1} = 1_E$$

so $V \in Un\, F_{n+1}$. Moreover

$$VA_{n+1}P = A_{n+1}X\,, \qquad A_{n+1}XV^* = A_{n+1}Q\,.$$

Put

$$U := A_{n+2}V + B_{n+2}V^*\,.$$

By Proposition 6.2.5 d), $U \in Un_0\, F_{n+2}$. We have

$$U(A_{n+2}A_{n+1}P)U^* = (A_{n+2}V + B_{n+2}V^*)A_{n+2}A_{n+1}P(A_{n+2}V^* + B_{n+2}V) =$$

$$= A_{n+2}A_{n+1}X(A_{n+2}V^* + B_{n+2}V) = A_{n+2}A_{n+1}Q\,.$$ ∎

PROPOSITION 6.2.7 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$ and $a \in Ker\, K_0(\varphi)$.*

a) There are $n \in \mathbb{N}$, $P \in Pr\check{F}_{\rightarrow n}$, and $U \in Un_0\,\check{G}_{\rightarrow,n+2}$ such that

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0 \qquad\qquad U(\check{\varphi}_{\rightarrow} P)U^* = \sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} P\,.$$

b) If φ is surjective then there is a $P \in Pr\check{F}_{\rightarrow}$ such that

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0, \qquad\qquad \check{\varphi}_{\rightarrow} P = \sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} P\,.$$

a) By Proposition 6.2.4, there are $m \in \mathbb{N}$ and $Q \in Pr\check{F}_{\rightarrow,m-1}$ such that

$$a = [Q]_0 - [\sigma^F_{\rightarrow} Q]_0\,.$$

Since $\check{\varphi} \circ \sigma^F = \sigma^G \circ \check{\varphi}$, by Proposition 6.1.10 c) and Corollary 6.1.11,

$$0 = K_0(\varphi)a = [\check{\varphi}_{\rightarrow} Q]_0 - [\check{\varphi}_{\rightarrow}\sigma^F_{\rightarrow} Q]_0 = [\check{\varphi}_{\rightarrow} Q]_0 - [\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} Q]_0\,.$$

By Corollary 6.1.6 a$\Rightarrow$c, there is an $n \in \mathbb{N}$, $n > m$, such that

$$\check{\varphi}_{\rightarrow} Q + (B_n)_{\rightarrow} \sim_0 \sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} Q + (B_n)_{\rightarrow} = \sigma^G_{\rightarrow}(\check{\varphi}_{\rightarrow} Q + (B_n)_{\rightarrow})\,.$$

Put

$$P := Q + (B_n)_{\rightarrow} \in Pr\check{F}_{\rightarrow n}\,.$$

Then

$$[P]_0 - [\sigma^F_{\rightarrow} P]_0 = [Q]_0 + [(B_n)_{\rightarrow}]_0 - [\sigma^F_{\rightarrow} Q]_0 - [(B_n)_{\rightarrow}]_0 = a\,,$$

$$[\check{\varphi}_{\rightarrow} P]_0 - [\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} P]_0 = [\check{\varphi}_{\rightarrow} Q]_0 + [(B_n)_{\rightarrow}]_0 - [\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} Q]_0 - [(B_n)_{\rightarrow}]_0 = 0\,.$$

By Corollary 6.1.6 a$\Rightarrow$b and Proposition 6.2.6, there is a $U \in Un_0\,\check{G}_{\rightarrow,n+2}$ with

$$U(\check{\varphi}_{\rightarrow} P)U^* = \sigma^G_{\rightarrow}\check{\varphi} P\,.$$

b) By a), there are $n \in \mathbb{N}$, $n > 2$, $Q \in Pr\check{F}_{\rightarrow,n-2}$, and $U \in Un_0\,\check{G}_{\rightarrow n}$ such that

$$a = [Q]_0 - [\sigma^F_{\rightarrow} Q]_0, \qquad\qquad U(\check{\varphi}_{\rightarrow} Q)U^* = \sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} Q\,.$$

Since $\varphi_n : \check{F}_n \longrightarrow \check{G}_n$ is surjective, by [4] Lemma 2.1.7 (i), there is a $V \in Un\,\check{F}_{\rightarrow n}$ with $\check{\varphi}_n V = U$. We put

$$P := VQV^* \sim_0 Q$$

so

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0$$

and

$$\check{\varphi}_{\to}P = (\check{\varphi}_{\to}V)(\check{\varphi}_{\to}Q)(\check{\varphi}_{\to}V^*) = U(\check{\varphi}_{\to}Q)U^* = \sigma^G_{\to}\check{\varphi}_{\to}Q\,,$$

$$\sigma^G_{\to}\check{\varphi}_{\to}P = \sigma^G_{\to}\check{\varphi}_{\to}Q = \check{\varphi}_{\to}P\,. \qquad \blacksquare$$

PROPOSITION 6.2.8 *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) $\check{\varphi}_{\to}$ *is injective.*

b) The following are equivalent for all $X \in \check{G}_{\to}$*:*

b_1) $X \in Im\,\check{\varphi}_{\to}$.

b_2) $\check{\psi}_{\to}X = \sigma^H_{\to}\check{\psi}_{\to}X$.

c) $K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)$ *is exact.*

a) $\check{\varphi}$ is injective (Proposition 4.1.5 a)) and the assertion follows from Proposition 6.1.10 b).

$b_1 \Rightarrow b_2$ follows from $\psi \circ \varphi = 0$.

$b_2 \Rightarrow b_1$. Let $n \in \mathbb{N}$ such that $X \in \check{G}_{\to n}$, which we identify with $\check{G}_n$. Then X has the form

$$X = \sum_{t \in T_n} ((\alpha_t, Y_t) \otimes id_K)V_t^{\check{G}}\,,$$

where $(\alpha_t, Y_t) \in \check{G}$ for every $t \in T_n$, and so by b_2),

$$\sum_{t \in T_n} ((\alpha_t, \psi Y_t) \otimes id_K)V_t^{\check{H}} = \check{\psi}_n X = \sigma^H_n \check{\psi}_n X = \sum_{t \in T_n} ((\alpha_t, 0) \otimes id_K)V_t^{\check{H}}\,.$$

It follows $\psi Y_t = 0$ for every $t \in T_n$ ([2] Theorem 2.1.9 a)). Thus for every $t \in T_n$ there is a $Z_t \in F$ with $\varphi Z_t = Y_t$ and we get

$$X = \sum_{t \in T_n} ((\alpha_t, \varphi Z_t) \otimes id_K)V_t^{\check{G}} =$$

$$= \check{\varphi}_n \left(\sum_{t\in T_n} ((\alpha_t, Z_t)\otimes id_K) V_t^{\check{F}} \right) \in Im\,\check{\varphi}_n \subset Im\,\check{\varphi}_{\rightarrow} \,.$$

c) By Corollary 6.2.3 a) and Proposition 6.2.2 e),

$$K_0(\psi)\circ K_0(\varphi) = K_0(\psi\circ\varphi) = 0$$

so $Im\,K_0(\varphi) \subset Ker\,K_0(\psi)$. Let $a \in Ker\,K_0(\psi)$. By Proposition 6.2.7 b), there is a $P \in Pr\,\check{G}_{\rightarrow}$ such that

$$a = [P]_0 - [\sigma^G_{\rightarrow} P]_0, \qquad\qquad \check{\psi}_{\rightarrow} P = \sigma^H_{\rightarrow}\check{\psi}_{\rightarrow} P\,.$$

Then P has the form

$$P = \sum_{t\in T_n} ((\alpha_t, X_t)\otimes id_K) V_t^{\check{G}}$$

for some $n \in \mathbb{N}$ with $(\alpha_t, X_t) \in E\times G$ for every $t \in T_n$, where we identified $\check{G}_n$ with $\check{G}_{\rightarrow n}$. We get

$$\sum_{t\in T_n} ((\alpha_t, \psi X_t)\otimes id_K) V_t^{\check{H}} = \check{\psi}_{\rightarrow} P = \sigma^H_{\rightarrow}\check{\psi}_{\rightarrow} P = \sum_{t\in T_n} ((\alpha_t, 0)\otimes id_K) V_t^{\check{H}} \,.$$

Thus $\psi X_t = 0$ ([2] Theorem 2.1.9 a)) and there is an $Y_t \in F$ with $\varphi Y_t = X_t$ for every $t \in T_n$. We put

$$Q := \sum_{t\in T_n} ((\alpha_t, Y_t)\otimes id_K) V_t^{\check{F}} \in Pr\,\check{F}_{\rightarrow}$$

with the usual identification ($\check{\varphi}$ is an embedding !). Then

$$\check{\varphi}_{\rightarrow} Q = \sum_{t\in T_n} ((\alpha_t, \varphi Y_t)\otimes id_K) V_t^{\check{G}} = \sum_{t\in T_n} ((\alpha_t, X_t)\otimes id_K) V_t^{\check{G}} = P$$

and by Proposition 6.2.2 c) (since $\check{\varphi}\circ\sigma^F = \sigma^G\circ\check{\varphi}$),

$$K_0(\varphi)([Q]_0 - [\sigma^F_{\rightarrow} Q]_0) = [\check{\varphi}_{\rightarrow} Q]_0 - [\check{\varphi}_{\rightarrow}\sigma^F_{\rightarrow} Q]_0 =$$

$$= [\check{\varphi}_{\rightarrow} Q]_0 - [\sigma^G_{\rightarrow}\check{\varphi}_{\rightarrow} Q]_0 = [P]_0 - [\sigma^G_{\rightarrow} P]_0 = a\,.$$

Thus $Ker\,K_0(\psi) \subset Im\,K_0(\varphi)$, $Ker\,K_0(\psi) = Im\,K_0(\varphi)$. ■

PROPOSITION 6.2.9 (Split Exact Theorem for K_0) *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\xleftarrow[\lambda]{}}{\xrightarrow{\psi}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ *then*

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \underset{\xleftarrow[K_0(\lambda)]{}}{\xrightarrow{K_0(\psi)}} K_0(H) \longrightarrow 0$$

is also split exact. In particular the map

$$K_0(F) \times K_0(H) \longrightarrow K_0(G), \quad (a,b) \longmapsto K_0(\varphi)a + K_0(\lambda)b$$

is a group isomorphism and $K_0(\check{F}) \approx K_0(E) \times K_0(F)$ *for every* E*-*C^**-algebra* F*.*

By Proposition 6.2.8 c), the second sequence is exact at $K_0(G)$. From

$$K_0(\psi) \circ K_0(\lambda) = K_0(\psi \circ \lambda) = K_0(id_H) = id_{K_0(H)}$$

(Corollary 6.2.3 a) and Proposition 6.2.2 d)) it follows that this sequence is (split) exact at $K_0(H)$.

Let $a \in Ker\, K_0(\varphi)$. By Proposition 6.2.7 a), there are $n \in \mathbb{N}$, $P \in Pr\,\check{F}_{\rightarrow n}$, and $U \in Un_0\, \check{G}_{\rightarrow, n+2}$ such that

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0, \qquad U(\check{\varphi}_{\rightarrow} P)U^* = \sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P\ .$$

Put

$$V := (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)U \in Un\, \check{G}_{\rightarrow, n+2}\ .$$

Then

$$\check{\psi}_{\rightarrow} V = (\check{\psi}_{\rightarrow} U^*)(\check{\psi}_{\rightarrow} U) = 1_{\rightarrow, n+2}, \qquad \sigma^H_{\rightarrow} \check{\psi}_{\rightarrow} V = \check{\psi}_{\rightarrow} V\ .$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $W \in Un\, \check{F}_{\rightarrow, n+2}$ with $\check{\varphi}_{\rightarrow} W = V$ ($\check{\varphi}$ is an embedding). We have

$$\check{\varphi}_{\rightarrow}(WPW^*) = V(\check{\varphi}_{\rightarrow} P)V^* = (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)U(\check{\varphi}_{\rightarrow} P)U^*(\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U) =$$

$$= (\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U^*)(\sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P)(\check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} U) = \check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow}(U^*(\sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P)U) =$$

$$= \check{\lambda}_{\rightarrow} \check{\psi}_{\rightarrow} \check{\varphi}_{\rightarrow} P = \sigma^G_{\rightarrow} \check{\varphi}_{\rightarrow} P = \check{\varphi}_{\rightarrow} \sigma^F_{\rightarrow} P\ .$$

Since $\check{\varphi}_{\rightarrow}$ is injective (Proposition 6.2.8 a)),

$$P \sim_0 WPW^* = \sigma^F_{\rightarrow} P, \qquad a = 0$$

and $K_0(\varphi)$ is injective.

The last assertion follows since

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\rightarrow}} E \longrightarrow 0$$

is a split exact sequence. ■

COROLLARY 6.2.10 *Let F, G be E-C*-algebras.*

a) If we put

$$\iota_1 : F \longrightarrow F \times G, \quad x \longmapsto (x,0), \qquad \pi_1 : F \times G \longrightarrow F, \quad (x,y) \longmapsto x,$$

$$\iota_2 : G \longrightarrow F \times G, \quad y \longmapsto (0,y), \qquad \pi_2 : F \times G \longrightarrow F, \quad (x,y) \longmapsto y,$$

then the sequences

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota_1)} K_0(F \times G) \underset{\overleftarrow{K_0(\iota_2)}}{\xrightarrow{K_0(\pi_2)}} K_0(G) \longrightarrow 0,$$

$$0 \longrightarrow K_0(G) \xrightarrow{K_0(\iota_2)} K_0(F \times G) \underset{\overleftarrow{K_0(\iota_1)}}{\xrightarrow{K_0(\pi_1)}} K_0(F) \longrightarrow 0$$

are split exact.

b) The map

$$K_0(F) \times K_0(G) \longrightarrow K_0(F \times G), \quad (a,b) \longmapsto K_0(\iota_1)a + K_0(\iota_2)b$$

is a group isomorphism **(Product Theorem for K_0)**.

a) is easy to see.

b) follows from a) and Proposition 6.2.9. ■

THEOREM 6.2.11 (Homotopy invariance of K_0)

a) If $\varphi, \psi : F \longrightarrow G$ are homotopic morphisms in $\mathfrak{M}_E$, then $K_0(\varphi) = K_0(\psi)$.

b) If $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ is a homotopy in $\mathfrak{M}_E$ then

$$K_0(\varphi) \circ K_0(\psi) = id_{K_0(G)}, \qquad K_0(\psi) \circ K_0(\varphi) = id_{K_0(F)}.$$

c) If F and G are homotopic E-C-algebras then $K_0(F)$ and $K_0(G)$ are isomorphic.*

d) If F is an E-C-algebra such that id_F is homotopic to*

$$0_F : F \longrightarrow F, \quad x \longmapsto 0$$

then F is homotopic to $\{0\}$.

e) If the E-C-algebra F is homotopic to $\{0\}$ then $K_0(F) = \{0\}$.*

a) Let

$$\phi_s : F \longrightarrow G, \qquad s \in [0,1]$$

be a pointwise continuous path of morphisms in $\mathfrak{M}_E$ such that $\phi_0 = \varphi$, $\phi_1 = \psi$. Then

$$\check{\phi}_s : \check{F} \longrightarrow \check{G}, \qquad s \in [0,1]$$

is a pointwise continuous path of morphisms in $\mathfrak{C}_E$ with $\check{\phi}_0 = \check{\varphi}$, $\check{\phi}_1 = \check{\psi}$ and for every $n \in \mathbb{N}$,

$$(\check{\phi}_s)_{\rightarrow n} : (\check{F})_{\rightarrow n} \longrightarrow (\check{G})_{\rightarrow n}, \qquad s \in [0,1]$$

is a pointwise continuous path in $\mathfrak{C}_E$ with $(\check{\phi}_0)_{\rightarrow n} = (\check{\varphi})_{\rightarrow n}$ and $(\check{\phi}_1)_{\rightarrow n} = (\check{\psi})_{\rightarrow n}$. For every $P \in Pr\check{F}_{\rightarrow n}$,

$$[0,1] \longrightarrow Pr(\check{G})_{\rightarrow n}, \quad s \longmapsto (\check{\phi}_s)_{\rightarrow n} P$$

is continuous so (by [4] Proposition 2.2.7)

$$K_0(\varphi)[P]_0 = [\varphi_{\rightarrow} P]_0 = [\psi_{\rightarrow} P]_0 = K_0(\psi)[P]_0$$

(Proposition 6.2.2 c)). By Proposition 6.2.4, $K_0(\varphi) = K_0(\psi)$.

b) follows from a), Corollary 6.2.3 a), and Proposition 6.2.2 d).

c) follows from b).

d) If we put $\varphi : F \longrightarrow \{0\}$ and $\psi : \{0\} \longrightarrow F$ then $\psi \circ \varphi = 0_F$ is homotopic to id_F and $\varphi \circ \psi$ is homotopic to $id_{\{0\}}$, so F is homotopic to $\{0\}$.

e) follows from c). ■

We show now that K_0 is continuous with respect to inductive limits.

THEOREM 6.2.12 (Continuity of K_0) *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in $\mathfrak{M}_E$ and let $\{F, (\varphi_i)_{i\in I}\}$ be its inductive limit in $\mathfrak{M}_E$. By* Corollary 6.2.3 a),

$$\{(K_0(F_i))_{i\in I}, (K_0(\varphi_{ij}))_{i,j\in I}\}$$

is an inductive system in the category of additive groups. Let $\{\mathscr{G}, (\psi_i)_{i\in I}\}$ be its limit in this category and let $\psi : \mathscr{G} \longrightarrow K_0(F)$ be the group homomorphism such that $\psi \circ \psi_i = K_0(\varphi_i)$ for every $i \in I$. Then ψ is a group isomorphism.

$\{(\check{F}_i)_{i\in I}, (\check{\varphi}_{ij})_{i,j\in I}\}$ is an inductive system in $\mathfrak{C}_E$ and by [2] Proposition 1.2.9 b), $\{\check{F}, (\check{\varphi}_i)_{i\in I}\}$ may be identified with its inductive limit in $\mathfrak{C}_E$. By[2] Proposition 2.3.5, for every $n \in \mathbb{N}$, $\{((\check{F}_i)_{\rightarrow n})_{i\in I}, ((\check{\varphi}_{ij})_{\rightarrow n})_{i,j\in I}\}$ is an inductive system in $\mathfrak{C}_E$ and $\{(\check{F}_{\rightarrow n}, ((\check{\varphi}_i)_{\rightarrow n})_{i\in I}\}$ may be identified with its inductive limit in $\mathfrak{C}_E$.

Step 1 ψ is surjective

Let $Q \in Pr(\check{F})_{\rightarrow n}$. By [5] L.2.2, there are $i \in I$ and $P \in Pr(\check{F}_i)_{\rightarrow n}$ such that $\|(\check{\varphi}_i)_{\rightarrow n}P - Q\| < 1$, so by [4] Proposition 2.2.4, $(\check{\varphi}_i)_{\rightarrow n}P \sim_0 Q$. By Proposition 6.2.2 b),c)

$$\psi\psi_i[P]_0 = K_0(\varphi_i)[P]_0 = K_0(\check{\varphi}_i)[P]_0 = [(\check{\varphi}_i)_{\rightarrow n}P]_0 = [Q]_0\,.$$

Since

$$Pr\check{F}_{\rightarrow} = \bigcup_{n\in\mathbb{N}} Pr(\check{F})_{\rightarrow n}\,,$$

ψ is surjective.

Step 2 ψ is injective

Let $a \in \mathscr{G}$ with $\psi a = 0$. Since $\mathscr{G} = \bigcup_{i\in I} Im\,\psi_i$, there is an $i \in I$ and an $a_i \in K_0(F_i)$ with $a = \psi_i a_i$. There are $n \in \mathbb{N}$ and $P, Q \in Pr(\check{F}_i)_{\rightarrow n}$ such that

$$a_i = [P]_0 - [Q]_0$$

(by Proposition 6.1.5 c)). By Proposition 6.2.2 c),

$$0 = \psi a = \psi\psi_i a = K_0(\varphi_i)a = K_0(\varphi_i)[P]_0 - K_0(\varphi_i)[Q]_0 =$$

$$= [(\check{\varphi}_i)_{\to n}P]_0 - [(\check{\varphi}_i)_{\to n}Q]_0 \,.$$

By Corollary 6.1.6 a⇒b, there is an $R \in Pr\,(\check{F}_i)_{\to}$ such that

$$PR = QR = 0, \qquad\qquad P+R \sim_0 Q+R$$

and we get

$$a = [P]_0 + [R]_0 - [Q]_0 - [R]_0 = [P+R]_0 - [Q+R]_0 = 0\,. \qquad \blacksquare$$

6.3 Stability of K_0

The stability of K_0 holds only under strong supplementary hypotheses. We present below such possible hypotheses, which we fix for this section. We shell give only a sketch of the proof.

Let S be a finite group, $\chi : \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow S$ an injective group homomorphism,

$$a := \omega(1,0), \qquad b := \omega(0,1), \qquad c := \omega(1,1)\,,$$

and g a Schur E-function for S such that

$$g(a,b) = g(a,c) = g(b,c) = -g(b,a) = 1_E\,.$$

We put for every $n \in \mathbb{N}$,

$$T_n := S^n = \left\{\, t \in S^{\mathbb{N}} \;\middle|\; m \in \mathbb{N}, m > n \Rightarrow t_m = 1 \,\right\}\,,$$

$$T := \bigcup_{n\in\mathbb{N}} T_n = \left\{\, t \in S^{\mathbb{N}} \;\middle|\; \{n \in \mathbb{N}, t_n \neq 1\} \text{is finite} \,\right\}\,,$$

$$f : T \times T \longrightarrow E\,, \quad (s,t) \longmapsto \prod_{n\in\mathbb{N}} g(s_n, t_n)\,,$$

$$\overset{n}{s} : \mathbb{N} \longrightarrow S\,, \quad m \longmapsto \begin{cases} s & \text{if} \quad m = n \\ 1 & \text{if} \quad m \neq n \end{cases}\,,$$

for every $s \in S$, and

$$C_n := \frac{1}{2}(V^f_{\overset{n}{a}} + V^f_{\overset{n}{b}}), \qquad A_n := C_n^* C_n, \qquad B_n := C_n C_n^*\,.$$

Then f is a Schur E-function for T and the following hold for all $s,t \in S$ and $n \in \mathbb{N}$:

$$f(\overset{n}{\bar{s}},\overset{n}{\bar{t}}) = g(s,t)\,,$$

$$\overset{n}{\bar{t}} = 1 \implies V^f_{\overset{n}{\bar{s}}} V^f_{\overset{n}{\bar{t}}} = V^f_{\overset{n}{\bar{t}}} V^f_{\overset{n}{\bar{s}}}\,,$$

$$s \in T_{n-1} \implies V^f_s V_{\overset{n}{\bar{t}}} = V_{\overset{n}{\bar{t}}} V^f_s\,,$$

$$A_n = \frac{1}{2}(V^f_1 + V^f_{\overset{n}{\bar{c}}}) \in Pr E_n, \qquad B_n = \frac{1}{2}(V^f_1 - V^f_{\overset{n}{\bar{c}}}) \in Pr E_n\,,$$

$$A_n + B_n = V^f_1 = 1_E\,,$$

so the assumptions of Axiom 5.0.3 are fulfilled.

Remark. If χ is bijective and $E = \mathbb{C}$ then the corresponding projective K-theory coincides with the usual K-theory.

PROPOSITION 6.3.1 *Let F be an full E-C*-algebra and $m,n \in \mathbb{N}$. We define*

$$\alpha := \alpha^F_{m,n} : (F_m)_n \longrightarrow F_{m+n}\,,$$

$$\beta := \beta^F_{m,n} : F_{m+n} \longrightarrow (F_m)_n\,,$$

by

$$(\alpha X)_{(s,t)} := (X_t)_s, \qquad ((\beta Y)_t)_s := Y_{(s,t)}$$

for every $X \in (F_m)_n$, $Y \in F_{m+n}$, and $(s,t) \in S^m \times S^n = S^{m+n}$, where the identification is given by the bijective map

$$S^m \times S^n \longrightarrow S^{m+n}, \quad (s,t) \longmapsto (s_1,\cdots,s_m,t_1,\cdots,t_n)\,.$$

a) α and β are E-C-isomorphisms and $\alpha = \beta^{-1}$.*

b) $\alpha A_n = A_{m+n}$.

c) The diagram

$$\begin{array}{ccc} (F_m)_{n-1} & \xrightarrow{\alpha^F_{m,n-1}} & F_{m+n-1} \\ {\scriptstyle \bar{\rho}^{F_m}_n}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\rho}^F_{m+n}} \\ (F_m)_n & \xrightarrow[\alpha^F_{m,n}]{} & F_{m+n} \end{array}$$

is commutative.

It is obvious that α and β are E-linear and $\alpha \circ \beta = id_{F_{m+n}}$, $\beta \circ \alpha = id_{(F_m)_n}$. Thus α and β are bijective and $\alpha = \beta^{-1}$.

For $X, Y \in (F_m)_n$ and $(s,t) \in S^m \times S^n$, by [2] Theorem 2.1.9 c),g),

$$(\alpha X^*)_{(s,t)} = ((X^*)_t)_s = (\tilde{f}(t)(X_{t^{-1}})^*)_s = \tilde{f}(s)\tilde{f}(t)((X_{t^{-1}})_{s^{-1}})^* =$$

$$= \tilde{f}((s,t))(\alpha X_{(s,t)^{-1}})^* = ((\alpha X)^*)_{(s,t)},$$

$$((\alpha X)(\alpha Y))_{(s,t)} =$$

$$= \sum_{(u,v)\in S^m \times S^n} f((u,v),(u^{-1}s, v^{-1}t))(\alpha X)_{(u,v)}(\alpha Y)_{(u^{-1}s, v^{-1}t)} =$$

$$= \sum_{(u,v)\in S^m \times S^n} f(u, u^{-1}s) f(v, v^{-1}t)(X_v)_u (Y_{v^{-1}t})_{u^{-1}s} =$$

$$= \sum_{v \in S^n} f(v, v^{-1}t)(X_v Y_{v^{-1}t})_s =$$

$$= \left(\sum_{v \in S^n} f(v, v^{-1}t) X_v Y_{v^{-1}t} \right)_s = ((XY)_t)_s = (\alpha(XY))_{(s,t)}$$

so α is a C*-homomorphism and the assertion follows.

b) follows from the definition of A_n and A_{m+n}.

c) follows from b). ∎

PROPOSITION 6.3.2 *Let* $F \xrightarrow{\varphi} G$ *be a morphism in* $\mathfrak{C}_E$ *and* $m, n \in \mathbb{N}$. *With the notation of* Proposition 6.3.1 *the diagram*

$$\begin{array}{ccc} (F_m)_n & \xrightarrow{\alpha^F_{m,n}} & F_{m+n} \\ {\scriptstyle (\varphi_m)_n}\downarrow & & \downarrow{\scriptstyle \varphi_{m+n}} \\ (G_m)_n & \xrightarrow[\alpha^G_{m,n}]{} & G_{m+n} \end{array}$$

is commutative.

For $X \in (F_m)_n$ and $(s,t) \in S^m \times S^n = S^{m+n}$,

$$(\varphi_{m+n}\alpha^F_{m,n}X)_{(s,t)} = \varphi(\alpha^F_{m,n}X)_{(s,t)} = \varphi(X_t)_s =$$

$$= (\varphi_m X_t)_s = (((\varphi_m)_n X)_t)_s = (\alpha^G_{m,n}(\varphi_m)_n X)_{(s,t)}$$

so

$$\varphi_{m+n} \circ \alpha^F_{m,n} = \alpha^G_{m,n} \circ (\varphi_m)_n \ . \qquad \blacksquare$$

THEOREM 6.3.3 (Stability for K_0) *If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ and $n \in \mathbb{N}$ then*

$$K_0(F_n) \approx K_0(F), \qquad K_0(G_n) \approx K_0(G), \qquad K_0(\varphi_n) \approx K_0(\varphi) \ . \qquad \blacksquare$$

Remark. If $(F_\infty, (\rho^F_n)_{n\in\mathbb{N}})$ and $(G_\infty, (\rho^G_n)_{n\in\mathbb{N}})$ denote the inductive limits in $\mathfrak{M}_E$ of the corresponding inductive systems $((F_n)_{n\in\mathbb{N}}, (\rho^F_{n,m})_{n,m\in\mathbb{N}})$ and $((G_n)_{n\in\mathbb{N}}, (\rho^G_{n,m})_{n,m\in\mathbb{N}})$ then, with obvious notation,

$$K_0(F_\infty) \approx K_0(F), \qquad K_0(G_\infty) \approx K_0(G), \qquad K_0(\varphi_\infty) \approx K_0(\varphi) \ .$$

Chapter 7

The Functor K_1

7.1 Definition of K_1

PROPOSITION 7.1.1 *If F is a full E-C^*-algebra and $n \in \mathbb{N}$ then*

$$\bar{\tau}_n^F : Un\, F_{n-1} \longrightarrow Un\, F_n\,, \quad U \longmapsto A_n U + B_n$$

is an injective group homomorphism with

$$\bar{\tau}_n^F (Un_{E_{n-1}}\, F_{n-1}) \subset Un_{E_n}\, F_n\,.$$

For $U,V \in Un\, F_n$ we put $U \sim_1 V$ if $UV^*, U^*V \in Un\, E_n$. $\sim_1$ is an equivalence relation and $\sim_h$ implies $\sim_1$.

For $U,V \in Un\, F_{n-1}$,

$$\bar{\tau}_n^F U^* = A_n U^* + B_n = (\bar{\tau}_n^F U)^*\,,$$

$$(\bar{\tau}_n^F U)(\bar{\tau}_n^F V) = (A_n U + B_n)(A_n V + B_n) = A_n UV + B_n = \bar{\tau}_n^F (UV)\,,$$

$$(\bar{\tau}_n^F U)(\bar{\tau}_n^F U)^* = (\bar{\tau}_n^F U)^*(\bar{\tau}_n^F U) = A_n + B_n = 1_{F_n}\,,$$

i.e. $\bar{\tau}_n^F$ is well-defined and it is a group homomorphism. If $\bar{\tau}_n^F U = 1_{F_n}$ then

$$A_n U + B_n = \bar{\tau}_n^F U = 1_{F_n} = 1_E = A_n + B_n, \qquad A_n U = A_n\,,$$

so by Proposition 6.1.1 c), $U = 1_{F_{n-1}} = 1_E$ and $\bar{\tau}_n^F$ is injective.

The other assertions are obvious. ■

DEFINITION 7.1.2 *Let F be a full E-C^*-algebra. We put for all $m,n \in \mathbb{N}$, $m < n$,*

$$\tau_{n,m}^F := \bar{\tau}_n^F \circ \bar{\tau}_{n-1}^F \circ \cdots \circ \bar{\tau}_{m+1}^F : Un\, F_m \longrightarrow Un\, F_n\,.$$

Then $\{(Un\, F_n)_{n\in\mathbb{N}},\, (\tau_{n,m})_{m,n\in\mathbb{N}}\}$ is an inductive system of groups with injective maps. We denote by $\{un F,\, (\tau_n^F)_{n\in\mathbb{N}}\}$ its inductive limit. τ_n^F is injective for every $n \in \mathbb{N}$, so $(\tau_n^F(Un\, F_n))_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of $un F$, the union of which is $un F$. We put for every $n \in \mathbb{N}$ and $U \in Un\, F_n$,

$$Un\, F_{\leftarrow n} := \tau_n^F (Un\, F_n), \qquad U_{\leftarrow} := U_{\leftarrow n} := U_{\leftarrow n}^F := \tau_n^F U\,,$$

$$1_{\leftarrow n} := 1_{\leftarrow n}^F := \tau_n^F 1_{F_n} \,(= \tau_n^F 1_E)\,.$$

$(\tau_n^F (Un_{E_n}\, F_n))_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of $un F$; we denote by $un_E\, F$ their union.

We often identify $Un\, F_n$ with $Un\, F_{\leftarrow n}$.

PROPOSITION 7.1.3 *For $m,n \in \mathbb{N}$, $m < n$, and $U \in Un\, F_m$,*

$$\tau^F_{n,m}U = \left(\prod_{i=m+1}^{n} A_i\right)U + \left(1_E - \prod_{i=m+1}^{n} A_i\right) .$$

We prove this identity by induction with respect to n. The identity holds for $n := m+1$. Assume it holds for $n-1 \geq m$. Then

$$\tau^F_{n,m}U = \bar{\tau}^F_n \tau^F_{n-1,m}U = A_n \tau^F_{n-1,m}U + B_n =$$

$$= A_n\left(\left(\prod_{i=m+1}^{n-1} A_i\right)U + \left(1_E - \prod_{i=m+1}^{n-1} A_i\right)\right) + B_n =$$

$$= \left(\prod_{i=m+1}^{n} A_i\right)U + \left(1_E - \prod_{i=m+1}^{n} A_i\right) .$$ ∎

PROPOSITION 7.1.4 *Let F be a full E-C*algebra.*

a) If $U,V \in Un\, F_{n-1}$ for some $n \in \mathbb{N}$ then

$$\bar{\tau}^F_n(UV) \sim_h \bar{\tau}^F_n(VU), \qquad \bar{\tau}^F_n(UVU^*) \sim_h \bar{\tau}^F_n(V) .$$

b) $un_E\, F$ is a normal subgroup of $un\, F$ and $un\, F/un_E\, F$ is commutative.

c) For all $U,V \in un\, F$,

$$UV^* \in un_E\, F \Longleftrightarrow U^*V \in un_E\, F .$$

We put $U \sim_1 V$ if $UV^* \in un_E\, F$. $\sim_1$ is an equivalence relation.

a) By Proposition 6.2.5 a),b),

$$\bar{\tau}^F_n(UV) = A_nUV + B_n = (A_nU + B_n)(A_nV + B_n) \sim_h$$

$$\sim_h (A_nU + B_n)(A_n + B_nV) = A_nU + B_nV \sim_h A_nV + B_nU \sim_h \bar{\tau}^F_n(VU) .$$

It follows

$$\bar{\tau}_n^F(UVU^*) \sim_h \bar{\tau}_n^F(U^*UV) = \bar{\tau}_n^F(V)\,.$$

b) $un_E\,F$ is obviously a subgroup of $un\,F$. The other assertions follow from a).

c) Let $q : un\,F \to un\,F/un_E\,F$ be the quotient map. If $UV^* \in un_E\,F$ then by b),

$$q(UV^*) = q(U)q(V^*) = q(V^*)q(U) = q(V^*U)\,,$$

$$V^*U \in un_E\,F\,, \qquad\qquad U^*V = (V^*U)^* \in un_E\,F\,. \qquad \blacksquare$$

DEFINITION 7.1.5 *We denote for every E-C*-algebra F by $K_1(F)$ the additive group obtained from the commutative group $un\check{F}/un_E\,\check{F}$* (Proposition 7.1.4 b)) *by replacing the multiplication with the addition $\oplus$; by this the neutral element (which corresponds to 1_E) is denoted by* 0. *For every $U \in un\check{F}$ we denote by $[U]_1$ its equivalence class in $K_1(F)$.*

Remark. Let F be a full E-C*-algebra. By Proposition 4.1.2 d), $\check{F}$ is isomorphic to $E \times F$, so in this case we may define K_1 using F instead of $\check{F}$ (as we did for K_0).

PROPOSITION 7.1.6 *Let $F \xrightarrow{\varphi} G$ be a morphism in $\mathfrak{M}_E$.*

a) For $m,n \in \mathbb{N}$, $m < n$, the diagram

$$\begin{array}{ccc} Un\,\check{F}_m & \xrightarrow{\tau_{n,m}^{\check{F}}} & Un\,\check{F}_n \\ {\scriptstyle\check{\varphi}_m}\downarrow & & \downarrow{\scriptstyle\check{\varphi}_n} \\ Un\,\check{G}_m & \xrightarrow[\tau_{n,m}^{\check{G}}]{} & Un\,\check{G}_n \end{array}$$

is commutative. Thus there is a unique group homomorphism

$$\check{\varphi}_{\leftarrow} : un\check{F} \longrightarrow un\check{G}$$

such that

$$\check{\varphi}_{\leftarrow} \circ \tau_n^{\check{F}} = \tau_n^{\check{G}} \circ \check{\varphi}_n$$

for every $n \in \mathbb{N}$.

b) $\varphi_{\leftarrow}(un_E \check{F}) \subset un_E \check{G}$*; if* φ *is surjective then* $\varphi_{\leftarrow}(un_E \check{F}) = un_E \check{G}$.

c) There is a unique group homomorphism

$$K_1(\varphi) : K_1(F) \longrightarrow K_1(G)$$

such that

$$K_1(\varphi)[U]_1 = [\check{\varphi}_{\leftarrow} U]_1$$

for every $U \in un \check{F}$.

d) $K_1(id_F) = id_{K_1(F)}$.

e) $K_1(\{0\}) = \{0\}$.

a) It is sufficient to prove the assertion for $n = m+1$. For $U \in Un\, \check{F}_m$,

$$\tau_{n,m}^{\check{G}} \check{\varphi}_m U = A_n(\check{\varphi}_m U) + B_n = \check{\varphi}_n(A_n U + B_n) = \check{\varphi}_n \tau_{n,m}^{\check{F}} U \ .$$

b) Since $\check{\varphi}_n(Un_{E_n} \check{F}_n) \subset Un_{E_n} \check{G}_n$ for every $n \in \mathbb{N}$, it follows $\varphi_{\leftarrow}(un_E \check{F}) \subset un_E \check{G}$. If φ is surjective then by [4] Lemma 2.1.7 (iii), we may replace the above inclusion relation by $=$.

c) follows from a) and b).

d) is obvious.

e) follows from $un\, E = un_E\, E$. ■

DEFINITION 7.1.7 *An E-C*-algebra F is called* **K-null** *if*

$$K_0(F) = K_1(F) = 0 \ .$$

Let $F \xrightarrow{\varphi} G$ *be a morphism in* $\mathfrak{M}_E$. *We say that* φ *is* **K-null** *if*

$$K_0(\varphi) = K_1(\varphi) = 0 \ .$$

We say that φ **factorizes through null** *if there are morphisms* $F \xrightarrow{\varphi'} H \xrightarrow{\varphi''} G$ *in* $\mathfrak{M}_E$ *such that* $\varphi = \varphi'' \circ \varphi'$ *and H is K-null.*

PROPOSITION 7.1.8

a) If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ *are morphisms in* $\mathfrak{M}_E$ *then*

$$\check{\psi}_{\leftarrow} \circ \check{\varphi}_{\leftarrow} = (\check{\psi} \circ \check{\varphi})_{\leftarrow} = \left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow} , \qquad K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi) .$$

b) If $\varphi = 0$ *then* $K_1(\varphi) = 0$.

c) **(Homotopy invariance of K_1)** *If* $\varphi, \psi : F \longrightarrow G$ *are homotopic morphisms in* $\mathfrak{M}_E$ *then*

$$K_1(\varphi) = K_1(\psi) .$$

d) **(Homotopy invariance of K_1)** *If* $F \xrightarrow{\varphi} G \xrightarrow{\psi} F$ *is a homotopy in* $\mathfrak{M}_E$ *then*

$$K_1(\varphi) : K_1(F) \longrightarrow K_1(G), \qquad K_1(\psi) : K_1(G) \longrightarrow K_1(F)$$

are isomorphisms and $K_1(\psi) = K_1(\varphi)^{-1}$.

e) If the E-C-algebra F is homotopic to* $\{0\}$ *then F is K-null.*

f) If a morphism in $\mathfrak{M}_E$ *factorizes through null then it is K-null.*

a) Since

$$\check{\psi}_n \circ \check{\varphi}_n = (\check{\psi} \circ \check{\varphi})_n = \left(\widecheck{\psi \circ \varphi}\right)_n$$

for every $n \in \mathbb{N}$ we get

$$\check{\psi}_{\leftarrow} \circ \check{\varphi}_{\leftarrow} = (\check{\psi} \circ \check{\varphi})_{\leftarrow} = \left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow} .$$

For $U \in un\check{F}$, by Proposition 7.1.6 c),

$$K_1(\psi)K_1(\varphi)[U]_1 = K_1(\psi)[\check{\varphi}_{\leftarrow}U]_1 = [\check{\psi}_{\leftarrow}\check{\varphi}_{\leftarrow}U]_1 =$$

$$= [(\check{\psi} \circ \check{\varphi})_{\leftarrow}U]_1 = \left[\left(\widecheck{\psi \circ \varphi}\right)_{\leftarrow} U\right]_1 = K_1(\psi \circ \varphi)[U]_1 ,$$

so $K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi)$.

b) If we put $\vartheta : F \longrightarrow \{0\}$, $\iota : \{0\} \longrightarrow G$ then $\varphi = \iota \circ \vartheta$ and by a) and Proposition 7.1.6 e), $K_1(\varphi) = 0$.

c) Let

$$\phi_s : F \longrightarrow G, \qquad s \in [0,1]$$

be a pointwise continuous path of morphisms in $\mathfrak{M}_E$ with $\phi_0 = \varphi$ and $\phi_1 = \psi$. Let $n \in \mathbb{N}$. Then

$$(\check{\phi}_s)_n : \check{F}_n \longrightarrow \check{G}_n, \qquad s \in [0,1]$$

is a pointwise continuous path of E-C*-homomorphisms with $(\check{\phi}_0)_n = \check{\varphi}_n$ and $(\check{\phi}_1)_n = \check{\psi}_n$. For every $U \in Un\,\check{F}_n$, the map

$$\vartheta : [0,1] \longrightarrow Un\,\check{G}_n\,, \quad s \longmapsto (\check{\phi}_s)_n U$$

is continuous and $\vartheta(0) = \check{\varphi}_n U$, $\vartheta(1) = \check{\psi}_n U$, i.e. $\check{\varphi}_n U$ and $\check{\psi}_n U$ are homotopic in $Un\,\check{G}_n$. It follows

$$K_1(\varphi)[\tau_n^{\check{F}} U]_1 = K_1(\psi)[\tau_n^{\check{F}} U]_1\,,$$

which implies $K_1(\varphi) = K_1(\psi)$.

d) follows from c) and Proposition 7.1.6 d).

e) By d) and Proposition 7.1.6 e), $K_1(F) = \{0\}$. By the Homotopy invariance of K_0 (Theorem 6.2.11 e)), F is K-null.

f) follows immediately from a), e), and Corollary 6.2.3 a). ■

PROPOSITION 7.1.9 *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ *then*

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H)$$

is also exact.

Let $a \in Ker\,K_1(\psi)$ and let $U \in un\,\check{G}$ with $a = [U]_1$. By Proposition 7.1.6 c),

$$0 = K_1(\psi)a = [\check{\psi}_{\leftarrow} U]_1\,, \qquad \check{\psi}_{\leftarrow} U \in un_E\,\check{H}\,.$$

By Proposition 7.1.6 b), there is a $V \in un_E\, \check{G}$ with $\check{\psi}_{\leftarrow} V = \check{\psi}_{\leftarrow} U$. We put $W := UV^*$. By Proposition 7.1.4 c), $[W]_1 = a$ and so

$$\check{\psi}_{\leftarrow} W = (\check{\psi}_{\leftarrow} U)(\check{\psi}_{\leftarrow} V)^* = 1_E \ .$$

W has the form

$$W = \sum_{t \in T_n} ((\alpha_t, X_t) \otimes id_K) V_t^{\check{G}}$$

for some $n \in \mathbb{N}$, where $(\alpha_t, X_t) \in E \times G$ for every $t \in T_n$. We get

$$1_E = \check{\psi}_n W = \sum_{t \in T_n} ((\alpha_t, \psi X_t) \otimes id_K) V_t^{\check{H}}$$

and so by [2] Theorem 2.1.9 a), $\psi X_t = 0$ for every $t \in T_n$. For every $t \in T_n$, let $Y_t \in F$ with $\varphi Y_t = X_t$ and put

$$W' := \sum_{t \in T_n} ((\alpha_t, Y_t) \otimes id_K) V_t^{\check{F}} \ .$$

Since $\check{\varphi} : \check{F} \longrightarrow \check{G}$ is an embedding, $W' \in Un\, \check{F}_{\leftarrow n}$ and by Proposition 7.1.6 c),

$$K_1(\varphi)[W']_1 = [\check{\varphi}_n W']_1 = [W]_1 = a \ .$$

Thus $Ker K_1(\psi) \subset Im K_1(\varphi)$.

Let now $U \in un \check{F}_{\leftarrow}$. By Proposition 7.1.8 a),b),

$$K_1(\psi) K_1(\varphi)[U]_1 = K_1(\psi \circ \varphi)[U]_1 = K_1(0)[U]_1 = 0$$

so $Im K_1(\varphi) \subset Ker K_1(\psi)$. ■

PROPOSITION 7.1.10 *The following are equivalent for every full E-C*-algebra F.*

a) $K_1(F) = \{0\}$.

b) For every $n \in \mathbb{N}$ *and* $U \in Un\, F_n$ *there is an* $m \in \mathbb{N}$, $m > n$, *with* $\tau^F_{m,n} U \sim_h 1_E$ *in* $Un\, F_m$.

$a \Rightarrow b$ Since

$$(1_E, U) \in Un\, E_n \times Un\, F_n = Un\, (E_n \times F_n) = Un\, (E \times F)_n \, ,$$

it follows from Proposition 4.1.2 d), $(1_E, U - 1_E) \in Un\, \check{F}_n$. By a), there is an $m \in \mathbb{N}$, $m > n$, with

$$U_0 := (1_E, \tau^F_{m,n} U - 1_E) = \tau^{\check{F}}_{m,n}(1_E, U - 1_E) \in Un_{E_m} \check{F}_m \,.$$

Thus there is a continuous map

$$[0,1] \longrightarrow Un\, \check{F}_m, \quad s \longmapsto U_s$$

with $U_1 \in Un\, E_m\, (\subset Un\, \check{F}_m)$. We put

$$U'_s := U_s(\sigma^F_m U_s)^* \, (\in Un\, \check{F}_m)$$

for every $s \in [0,1]$. Then the map

$$[0,1] \longrightarrow Un\, \check{F}_m, \quad s \longmapsto U'_s$$

is continuous and $U'_0 = U_0$, $U'_1 = 1_E$. Let

$$\varphi : \check{F} \longrightarrow E \times F\,, \quad (\alpha, x) \longmapsto (\alpha, x + \alpha)$$

be the E-C*-isomorphism of Proposition 4.1.2 d). Then

$$U'' : [0,1] \longrightarrow Un\, E_n \times Un\, F_n\,, \quad s \longmapsto \varphi_m U'_s$$

is continuous and

$$U''_0 = \varphi_m U'_0 = (1_E, \tau^F_{m,n} U)\,, \qquad U''_1 = \varphi_m U'_1 = (1_E, 1_E)\,.$$

Thus $\tau^F_{m,n} U \sim_h 1_E$ in $Un\, F_m$.

$b \Rightarrow a$ Let $a \in K_1(F)$. There are $n \in \mathbb{N}$ and $U \in Un\, \check{F}_n$ with $a = [U]_1$. Since $U(\sigma^F_n U)^* \sim_1 U$, we may assume $U = U(\sigma^F_n U)^*$, i.e. $\sigma^F_n U = 1_E$. Thus there is a unique $X \in F_n$ with $\iota^F_n X = U - 1_E$. Then

$$U' := X + 1_E \in Un\, F_n\,.$$

By b), there is an $m \in \mathbb{N}$, $m > n$, with $\tau^F_{m,n} U' \sim_h 1_E$. By Proposition 4.1.2 d),

$$U = (1_E, X) = (1_E, U' - 1_E)\,, \qquad \tau^{\check{F}}_{m,n} U = (1_E, \tau^F_{m,n} U' - 1_E) \sim_h (1_E, 0)\,,$$

i.e. $a = [U]_1 = 0$. ■

COROLLARY 7.1.11 *If F is a finite-dimensional full E-C*-algebra then $K_1(F) = \{0\}$.*

For every $n \in \mathbb{N}$, F_n is finite-dimensional and so there is a finite family $(k_i)_{i\in I}$ in $\mathbb{N}$ such that $F_n \approx \prod_{i\in I} \mathbb{C}_{k_i,k_i}$. Thus every $U \in Un\, F_n$ is homotopic to 1_E in $Un\, F_n$. By Proposition 7.1.10 $b \Rightarrow a$, $K_1(F) = \{0\}$. ■

COROLLARY 7.1.12 *If the spectrum of E is totally disconnected (this happens e.g. if E is a W*-algebra* ([1] Corollary 4.4.1.10)*) then $Un\, E_n = Un_0\, E_n$ for every $n \in \mathbb{N}$ and so $K_1(E) = \{0\}$.*

Let Ω be the spectrum of E and let $U \in Un\, E_n$. U has the form

$$U = \sum_{t\in T_n} (U_t \otimes id_K)V_t\,,$$

with $U_t \in E$ for every $t \in T_n$. We put

$$U(\omega) := \sum_{t\in T_n} (U_t(\omega) \otimes id_K)V_t$$

for every $\omega \in \Omega$ and denote by $\sigma(U(\omega))$ its spectrum, which is finite. Let $\omega_0 \in \Omega$ and let $\theta_0 \in [0,2\pi[$ such that $e^{i\theta_0} \notin \sigma(U(\omega_0))$. By [1] Corollary 2.2.5.2, there is o clopen neighborhood Ω_0 of ω_0 such that $e^{i\theta_0}$ does not belong to the spectrum of $U(\omega)$ for all $\omega \in \Omega_0$. Assume for a moment $\Omega_0 = \Omega$ and put for every $s \in [0,1]$,

$$h_s : \mathbb{T} \setminus \{\alpha\} \longrightarrow \mathbb{T}, \quad e^{i\vartheta} \longmapsto e^{i\vartheta s}, \qquad\qquad W_s := h_s(U)\,,$$

where $\vartheta \in]\vartheta_0 - 2\pi, \vartheta_0[$. Then

$$[0,1] \longrightarrow Un\, E_n, \quad s \longmapsto W_s$$

is a continuous path in $Un\, E_n$ ([1] Corollaries 4.1.2.13 and 4.1.3.5) with $W_1 = U$ and $W_0 = 1_E$. Thus $U \in Un_0\, E_n$.

Since Ω is the union of a finite family of pairwise disjoint clopen sets of the above form Ω_0, $U \in Un_0\, E_n$.

By Proposition 7.1.10 $b \Rightarrow a$, $K_1(E) = \{0\}$. ■

7.2 The Index Map

Throughout this section

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

denotes an exact sequence in $\mathfrak{M}_E$ and $n \in \mathbb{N}$.

PROPOSITION 7.2.1 *Let $U \in Un\, \check{H}_{n-1}$.*

a) There are $V \in Un\, \check{G}_n$ and $P \in Pr\, \check{F}_n$ such that

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* \,.$$

b) If $W \in Un\, \check{G}_n$ and $Q \in Pr\, \check{F}_n$ such that

$$\check{\psi}_n W = A_n U + B_n U^*, \qquad \check{\varphi}_n Q = W A_n W^*$$

then $\sigma_n^F Q = A_n$ and $P \sim_0 Q$.

c) Let $U_0 \in Un\, \check{H}_{n-1}$, $V_0 \in Un\, \check{G}_n$, and $P_0 \in Pr\, \check{F}_n$ with

$$U_0 \sim_1 U, \qquad \check{\psi}_n V_0 = A_n U_0 + B_n U_0^*, \qquad \check{\varphi}_n P_0 = V_0 A_n V_0^* \,.$$

Then $P_0 \sim_0 P$.

d) If $U \in Un_{E_{n-1}}\, \check{H}_{n-1}$ then $P \sim_0 A_n$.

a) By Proposition 6.2.5 d), $A_n U + B_n U^* \in Un_0\, \check{H}_n$ so by [4] Lemma 2.1.7 (i) (and [2] Theorem 2.1.9 a)), there is a $V \in Un_0\, \check{G}_n$ with $\check{\psi}_n V = A_n U + B_n U^*$. We have

$$\check{\psi}_n (V A_n V^*) = (A_n U + B_n U^*) A_n (A_n U^* + B_n U) = A_n \,,$$

$$\sigma_n^H \check{\psi}_n (V A_n V^*) = \sigma_n^H A_n = A_n = \check{\psi}_n (V A_n V^*) \,,$$

so by Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $P \in Pr\, \check{F}_n$ with $\check{\varphi}_n P = V A_n V^*$.

b) Since $\pi^F = \pi^H \circ \check{\psi} \circ \check{\varphi}$, we have

$$\pi_n^F Q = \pi_n^H \check{\psi}_n \check{\varphi}_n Q = \pi_n^H \check{\psi}_n (W A_n W^*) =$$

$$= \pi_n^H((A_nU + B_nU^*)A_n(A_nU^* + B_nU)) = \pi_n^H A_n = A_n\,,$$

$\sigma_n^F Q = A_n$. Since

$$\check{\psi}_n(WV^*) = (A_nU + B_nU^*)(A_nU^* + B_nU) = A_n + B_n = 1_E = \sigma_n^H \check{\psi}_n(WV^*)\,,$$

by Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $Z \in Un\,\check{F}_n$ with $\check{\varphi}_n Z = WV^*$. Then

$$\check{\varphi}_n(ZPZ^*) = (WV^*)(VA_nV^*)(VW^*) = WA_nW^* = \check{\varphi}_n Q\,,$$

$$ZPZ^* = Q, \qquad\qquad P \sim_0 Q\,.$$

c) By Proposition 7.1.4 c), $U^*U_0, UU_0^* \in Un_{E_{n-1}}\,\check{H}_{n-1}$ so by [4] Lemma 2.1.7 (iii), there are $X,Y \in Un\,\check{G}_{n-1}$ such that

$$\check{\psi}_{n-1}X = U^*U_0, \qquad\qquad \check{\psi}_{n-1}Y = UU_0^*\,.$$

We put

$$Z := V(A_nX + B_nY)\,.$$

By Proposition 6.2.5 c), $Z \in Un\,\check{G}_n$. We have

$$\check{\psi}_n Z = (A_nU + B_nU^*)(A_nU^*U_0 + B_nUU_0^*) = A_nU_0 + B_nU_0^*\,,$$

$$\check{\psi}_n(ZA_nZ^*) = (A_nU_0 + B_nU_0^*)A_n(A_nU_0^* + B_nU_0) = A_n = \sigma_n^H \check{\psi}_n(ZA_nZ^*)\,.$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there is a $Q \in Pr\,\check{F}_n$ with $\check{\varphi}_n Q = ZA_nZ^*$. By b), $Q \sim_0 P_0$. From

$$\check{\varphi}_n Q = ZA_nZ^* = V(A_nX + B_nY)A_n(A_nX^* + B_nY^*)V^* = VA_nV^* = \check{\varphi}_n P$$

it follows $P_0 \sim_0 Q = P$ (by [2] Theorem 2.1.9 a)).

d) By c), we may take $U = 1_E$. Further we may take $W = 1_E$ and $Q = A_n$ in b), so $P \sim A_n$. ∎

PROPOSITION 7.2.2 *For every $i \in \{1,2\}$ let $U_i \in Un\,\check{H}_{n-1}$, $V_i \in Un\,\check{G}_n$, and $P_i \in Pr\,\check{F}_n$ such that*

$$\check{\psi}_n V_i = A_nU_i + B_nU_i^*, \qquad\qquad \check{\varphi}_n P_i = V_iA_nV_i^*\,.$$

Put

$$X := A_{n+1}A_n + C_{n+1}^*C_n + C_{n+1}C_n^* + B_{n+1}B_n, \qquad U := A_nU_1 + B_nU_2\,,$$

$$V := X(A_{n+1}V_1 + B_{n+1}V_2)X, \qquad\qquad P := X(A_{n+1}P_1 + B_{n+1}P_2)X\,,$$

a) $X \in Un_0\, E_{n+1},\ U \in Un\, \check{H}_n,\ V \in Un\, \check{G}_{n+1},\ P \in Pr\, \check{F}_{n+1}$.

b) $\check{\psi}_{n+1} V = A_{n+1} U + B_{n+1} U^*,\ \check{\varphi}_{n+1} P = V A_{n+1} V^*$.

a) We have

$$X^2 = A_{n+1}A_n + A_{n+1}B_n + B_{n+1}A_n + B_{n+1}B_n = 1_E \ .$$

Since X is selfadjoint it follows $X \in Un_0\, E_{n+1}$ ([4] Lemma 2.1.3 (ii)) and so $P \in Pr\, \check{F}_{n+1}$. By Proposition 6.2.5 c), $U \in Un\, \check{H}_n$ and $V \in Un\, \check{G}_{n+1}$.

b) We have

$$XA_{n+1}X = (A_{n+1}A_n + C_{n+1}C_n^*)X = A_{n+1}A_n + B_{n+1}A_n = A_n \, ,$$

$$XB_{n+1}X = (C_{n+1}^* C_n + B_{n+1}B_n)X = A_{n+1}B_n + B_{n+1}B_n = B_n \, ,$$

$$XA_nX = A_{n+1}, \qquad XB_nX = B_{n+1} \, ,$$

$$XA_{n+1}A_nX = A_{n+1}A_n, \qquad XA_{n+1}B_nX = B_{n+1}A_n \, ,$$

$$XB_{n+1}A_nX = A_{n+1}B_n, \qquad XB_{n+1}B_nX = B_{n+1}B_n \, ,$$

$$\check{\psi}_{n+1}V = X(A_{n+1}(A_nU_1 + B_nU_1^*) + B_{n+1}(A_nU_2 + B_nU_2^*))X =$$

$$= A_{n+1}A_nU_1 + B_{n+1}A_nU_1^* + A_{n+1}B_nU_2 + B_{n+1}B_nU_2^* = A_{n+1}U + B_{n+1}U^* \, ,$$

$$VA_{n+1}V^* = X(A_{n+1}V_1 + B_{n+1}V_2)XA_{n+1}X((A_{n+1}V_1^* + B_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1 + B_{n+1}V_2)A_n(A_{n+1}V_1^* + B_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1A_nA_{n+1}V_1^* + B_{n+1}V_2A_nB_{n+1}V_2^*)X =$$

$$= X(A_{n+1}V_1A_nV_1^* + B_{n+1}V_2A_nV_2^*)X =$$

$$= X(A_{n+1}\check{\varphi}_nP_1 + B_{n+1}\check{\varphi}_nP_2)X =$$

$$= \check{\varphi}_{n+1}(X(A_{n+1}P_1 + B_{n+1}P_2)X) = \check{\varphi}_{n+1}P \ . \qquad \blacksquare$$

COROLLARY 7.2.3 *There is a unique group homomorphism, called* **the index map***,*

$$\delta_1 : K_1(H) \longrightarrow K_0(F)$$

such that

$$\delta_1[U]_1 = [P]_0 - [\sigma^F_{\rightarrow} P]_0$$

for every $U \in un\check{H}$, where P satisfies the conditions of Proposition 7.2.1 a).

By Proposition 7.2.1 a),b), the map

$$\nu_n : Un\,\check{H}_{n-1} \longrightarrow K_0(F)\,, \quad U \longmapsto [P]_0 - [\sigma_n^F P]_0$$

is well-defined for every $n \in \mathbb{N}$, where P is associated to U as in Proposition 7.2.1 a). By Proposition 7.2.1 c), $\nu_n U = \nu_n U_0$ for all $U, U_0 \in Un\,\check{H}_{n-1}$ with $U \sim_1 U_0$. With the notation of Proposition 7.2.2,

$$\nu_{n+1}(A_n U_1 + B_n U_2) = \nu_{n+1} U = [P]_0 - [\sigma_{n+1}^F P]_0 =$$

$$= [A_{n+1} P_1 + B_{n+1} P_2]_0 - [\sigma_{n+1}^F (A_{n+1} P_1 + B_{n+1} P_2)]_0 =$$

$$= [P_1]_0 + [P_2]_0 - [\sigma_n^F P_1]_0 - [\sigma_n^F P_2]_0 = \nu_n U_1 + \nu_n U_2\,.$$

Thus by Proposition 7.2.1 d) (and Proposition 7.2.2), for $U \in Un\,\check{H}_{n-1}$,

$$\nu_{n+1}(\bar{\tau}_n^{\check{H}} U) = \nu_{n+1}(A_n U + B_n) = \nu_n U + \nu_n 1_E = \nu_n U\,.$$

Hence the map

$$\nu : un\check{H} \longrightarrow K_0(F)\,, \quad U \longmapsto \nu_n U$$

is well-defined, where $U \in Un\,\check{H}_{n-1}$ for some $n \in \mathbb{N}$. By Proposition 7.2.1 d), again, ν induces a map $\delta_1 : K_1(H) \longrightarrow K_0(F)$, which is additive by the above considerations. The uniqueness follows from the fact that the map $[\cdot]_1 : un\check{H} \longrightarrow K_1(H)$ is surjective. ■

PROPOSITION 7.2.4 *Let*

$$0 \longrightarrow F' \xrightarrow{\varphi'} G' \xrightarrow{\psi'} H' \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ *and* δ_1' *its associated index map. If the diagram in* $\mathfrak{M}_E$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H & \longrightarrow & 0 \\
 & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & F' & \xrightarrow[\varphi']{} & G' & \xrightarrow[\psi']{} & H' & \longrightarrow & 0
\end{array}$$

is commutative then the diagram

$$\begin{array}{ccc}
K_1(H) & \xrightarrow{\delta_1} & K_0(F) \\
{\scriptstyle K_1(\beta)}\downarrow & & \downarrow{\scriptstyle K_0(\gamma)} \\
K_1(H') & \xrightarrow[\delta_1']{} & K_0(F')
\end{array}$$

is also commutative.

Let $U \in Un\,\check{H}_{n-1}$, $V \in Un\,\check{G}_n$, and $P \in Pr\,\check{F}_n$ with

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* .$$

Put

$$V' := \check{\alpha}_n V \in Un\,\check{G}'_n, \qquad P' := \check{\gamma}_n P \in Pr\,\check{F}'_n .$$

Then

$$\check{\psi}'_n V' = \check{\psi}'_n \check{\alpha}_n V = \check{\beta}_n \check{\psi}_n V = A_n \check{\beta}_{n-1} U + B_n \check{\beta}_{n-1} U^* ,$$

$$\check{\varphi}'_n P' = \check{\varphi}'_n \check{\gamma}_n P = \check{\alpha}_n \check{\varphi}_n P = \check{\alpha}_n (V A_n V^*) = V' A_n V'^* .$$

By Corollary 7.2.3 for δ'_1, Proposition 7.1.6 c), and Proposition 6.2.2 c),

$$\delta'_1 K_1(\beta)[U]_1 = \delta'_1[\check{\beta}_{n-1}U]_1 = [P']_0 - [\sigma_n^{F'} P']_0 = [\check{\gamma}_n P]_0 - [\sigma_n^{F'} \check{\gamma}_n P]_0 =$$

$$= [\check{\gamma}_n P]_0 - [\check{\gamma}_n \sigma_n^F P]_0 = K_0(\gamma)([P]_0 - [\sigma_n^F P]_0) = K_0(\gamma)\delta_1[U]_1 . \qquad \blacksquare$$

PROPOSITION 7.2.5

a) $\delta_1 \circ K_1(\psi) = 0.$

b) $K_0(\varphi) \circ \delta_1 = 0.$

a) Let $U \in Un\,\check{G}_{n-1}$ and put

$$V := \bar{\tau}_n^{\check{G}} U = A_n U + B_n \in Un\,\check{G}_n .$$

Then

$$\check{\psi}_n V = A_n(\check{\psi}_{n-1} U) + B_n ,$$

$$(\check{\psi}_n V) A_n (\check{\psi}_n V)^* = (A_n(\check{\psi}_{n-1} U) + B_n) A_n (A_n(\check{\psi}_{n-1} U)^* + B_n) = A_n ,$$

so (by Proposition 7.1.6 c))

$$\delta_1 K_1(\psi)[U]_1 = \delta_1[\check{\psi}_{n-1} U]_1 = [A_n]_0 - [\sigma_n^F A_n]_0 = 0 .$$

b) Let $U \in Un\,\check{H}_{n-1}$, $V \in Un\,\check{G}_n$, and $P \in Pr\,\check{F}_n$ with

$$\check{\psi}_n V = A_n U + B_n U^*, \qquad \check{\varphi}_n P = V A_n V^* .$$

By Proposition 6.2.2 c) (since $\check{\varphi} \circ \sigma^F = \sigma^G \circ \check{\varphi}$),

$$K_0(\varphi)\delta_1[U]_1 = K_0(\varphi)([P]_0 - [\sigma_n^F P]_0) =$$

$$= [\check{\varphi}_n P]_0 - [\check{\varphi}_n \sigma_n^F P]_0 = [\check{\varphi}_n P]_0 - [\sigma_n^G \check{\varphi}_n P]_0 =$$

$$= [VA_nV^*]_0 - [(\sigma_n^G V)A_n(\sigma_n^G V)^*]_0 = [A_n]_0 - [A_n]_0 = 0 .$$ ■

PROPOSITION 7.2.6 *Let $U \in Un\,\check{H}_{n-1}$. There are $V \in \check{G}_n$ and $P,Q \in Pr\,\check{F}_n$ such that*

$$V^*V \in Pr\,\check{G}_n, \qquad \check{\psi}_n V = A_n U ,$$

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^*, \qquad \delta_1[U]_1 = [P]_0 - [Q]_0 .$$

By Proposition 6.2.5 d), $A_nU + B_nU^* \in Un_0\,\check{H}_n$. Since $\check{\psi}_n$ is surjective, by [4] Lemma 2.1.7 (i), there is a $V_0 \in Un\,\check{G}_n$ with $\check{\psi}_n V_0 = A_nU + B_nU^*$. Put $V := V_0A_n \in \check{G}_n$. Then

$$V^*V = A_nV_0^*V_0A_n = A_n \in Pr\,\check{G}_n$$

and

$$\check{\psi}_n V = (\check{\psi}_n V_0)A_n = (A_nU + B_nU^*)A_n = A_nU .$$

We have

$$\check{\psi}_n(1_E - V^*V) = 1_E - A_n = B_n = \check{\psi}_n(1_E - VV^*) .$$

By Proposition 6.2.8 $b_2 \Rightarrow b_1$, there are $P,Q \in Pr\,\check{F}_n$ with

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^* .$$

Put

$$W := A_{n+1}V + C_{n+1}(1_E - V^*V) + C_{n+1}^*(1_E - VV^*) + B_{n+1}V^* \in \check{G}_{n+1} ,$$

$$Z := A_n + (C_{n+1} + C_{n+1}^*)B_n \in E_{n+1} .$$

Since $VV^*V = V$, $V^*VV^* = V^*$, and

$$W^* = A_{n+1}V^* + C_{n+1}^*(1_E - V^*V) + C_{n+1}(1_E - VV^*) + B_{n+1}V ,$$

we get

$$WW^* = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) + A_{n+1}(1_E - VV^*) + B_{n+1}V^*V =$$

$$= A_{n+1} + B_{n+1} = 1_E \,,$$

$$W^*W = A_{n+1}V^*V + A_{n+1}(1_E - V^*V) + B_{n+1}(1_E - VV^*) + B_{n+1}VV^* =$$

$$= A_{n+1} + B_{n+1} = 1_E \ .$$

By Proposition 6.2.5 a),

$$Z^2 = A_n + B_n = 1_E$$

so $W \in Un\,\check{G}_{n+1}$, $Z \in Un\,E_{n+1}$, and $ZW \in Un\,\check{G}_{n+1}$. By the above and Proposition 6.2.5 a),

$$\check{\psi}_{n+1}W = A_{n+1}A_nU + (C_{n+1} + C_{n+1}^*)B_n + B_{n+1}A_nU^* \,,$$

$$\check{\psi}_{n+1}(ZW) = Z\check{\psi}_{n+1}W =$$

$$= (A_n + (C_{n+1} + C_{n+1}^*)B_n)(A_{n+1}A_nU + (C_{n+1} + C_{n+1}^*)B_n + B_{n+1}A_nU^*) =$$

$$= A_{n+1}A_nU + B_{n+1}A_nU^* + B_n = A_{n+1}A_nU + B_{n+1}A_nU^* + (A_{n+1} + B_{n+1})B_n =$$

$$= A_{n+1}(A_nU + B_n) + B_{n+1}(A_nU^* + B_n) \ .$$

We put

$$R := A_{n+1}(1_E - Q) + B_{n+1}P \in Pr\,\check{F}_{n+1} \ .$$

Using again $VV^*V = V$ and $V^*VV^* = V^*$,

$$\check{\varphi}_{n+1}R = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) \,,$$

$$WA_{n+1} = A_{n+1}V + C_{n+1}(1_E - V^*V) \,,$$

$$WA_{n+1}W^* = A_{n+1}VV^* + B_{n+1}(1_E - V^*V) = \check{\varphi}_{n+1}R \,,$$

$$ZWA_{n+1}W^*Z = Z(\check{\varphi}_{n+1}R)Z = \check{\varphi}_{n+1}(ZRZ) \ .$$

Since $ZRZ \sim_0 R$ and $U \sim_1 A_nU + B_n$, by the definition of δ_1,

$$\delta_1[U]_1 = \delta_1[A_nU + B_n]_1 = [R]_0 - [\sigma_{n+1}^F R]_0 \ .$$

Since $\pi^H \circ \check{\psi} \circ \check{\varphi} = \pi^F$, by the above,

$$\pi_n^F P = \pi_n^H \check{\psi}_n \check{\varphi}_n P = \pi_n^H \check{\psi}_n(1_E - V^*V) = \pi_n^H B_n = B_n = \pi_n^F Q \ .$$

Thus by Proposition 6.1.3 (and Proposition 7.2.1 b)),

$$\sigma_{n+1}^F R = A_{n+1}(1_E - B_n) + B_{n+1}B_n \sim_0 A_{n+1}B_n + A_{n+1}A_n =$$

$$= A_{n+1} = \bar{\rho}^{\check{F}}_{n+1} 1_E \sim_0 1_E$$

and we get

$$[R]_0 = [1_E - Q]_0 + [P]_0 = [1_E]_0 + [P]_0 - [Q]_0\,,$$

$$\delta_1[U]_1 = [1_E]_0 + [P]_0 - [Q]_0 - [1_E]_0 = [P]_0 - [Q]_0\,. \qquad \blacksquare$$

PROPOSITION 7.2.7 *Ker* $\delta_1 \subset Im\, K_1(\psi)$.

Let $a \in Ker\ \delta_1$ and let $U \in Un\, \check{H}_{n-1}$ with $a = [U]_1$. By Proposition 7.2.6, there are $V \in \check{G}_n$ and $P, Q \in Pr\, \check{F}_n$ such that $V^*V \in Pr\, \check{G}_n$, $\check{\psi}_n V = A_n U$,

$$\check{\varphi}_n P = 1_E - V^*V, \qquad \check{\varphi}_n Q = 1_E - VV^*, \qquad \delta_1[U]_1 = [P]_0 - [Q]_0\,.$$

Then $[P]_0 = [Q]_0$. By Corollary 6.1.6 a$\Rightarrow$c, there is an $m \in \mathbb{N}$, $m > n+1$, and an $X \in \check{F}_m$ such that

$$X^*X = \left(\prod_{i=n+1}^{m} A_i\right) P + \left(1_E - \prod_{i=n+1}^{m} A_i\right)\,,$$

$$XX^* = \left(\prod_{i=n+1}^{m} A_i\right) Q + \left(1_E - \prod_{i=n+1}^{m} A_i\right)\,.$$

Put $W := \check{\varphi}_m X$. Then

$$W^*W = \check{\varphi}_m(X^*X) = \left(\prod_{i=n+1}^{m} A_i\right)(1_E - V^*V) + \left(1_E - \prod_{i=n+1}^{m} A_i\right) =$$

$$= 1_E - \left(\prod_{i=n+1}^{m} A_i\right) V^*V\,,$$

$$WW^* = 1_E - \left(\prod_{i=n+1}^{m} A_i\right) VV^*\,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) VV^*WW^* = \left(\prod_{i=n+1}^{m} A_i\right) V^*VW^*W = 0\,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) V^*W = \left(\prod_{i=n+1}^{m} A_i\right) VW^* = 0\,,$$

$$\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)^* \left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right) =$$

$$= \left(\prod_{i=n+1}^{m} A_i\right) V^*V + W^*W = 1_E \,,$$

$$\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)\left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right)^* =$$

$$= \left(\prod_{i=n+1}^{m} A_i\right) VV^* + WW^* = 1_E \,,$$

$$\left(\prod_{i=n+1}^{m} A_i\right) V + W \in Un\,\check{G}_m \,.$$

From

$$\check{\psi}_m(W^*W) = 1_E - \left(\prod_{i=n+1}^{m} A_i\right) \check{\psi}_m(V^*V) =$$

$$= 1_E - \left(\prod_{i=n+1}^{m} A_i\right) A_n = \check{\psi}_m(WW^*) \,,$$

since $\check{\psi}_m W = \check{\psi}_m \check{\varphi}_m X \in E_m$, it follows

$$\check{\psi}_m W + \left(\prod_{i=n}^{m} A_i\right) \in Un\,E_m \,.$$

By the above,

$$\left(\prod_{i=n}^{m} A_i\right) U\,\check{\psi}_m W^* = \left(\prod_{i=n+1}^{m} A_i\right) (\check{\psi}_m V)(\check{\psi}_m W^*) =$$

$$= \check{\psi}_m \left(\left(\prod_{i=n+1}^{m} A_i\right) VW^*\right) = 0 \,,$$

$$(\check{\psi}_m W)^*(\check{\psi}_m W)\left(\prod_{i=n}^{m} A_i\right) = 0 \,, \qquad (\check{\psi}_m W)\left(\prod_{i=n}^{m} A_i\right) = 0 \,,$$

$$\check{\psi}_m \left(\left(\prod_{i=n+1}^{m} A_i\right) V + W\right) = \left(\prod_{i=n}^{m} A_i\right) U + \check{\psi}_m W \sim_1$$

$$\sim_1 \left(\left(\prod_{i=n}^{m} A_i\right) U + \check{\psi}_m W\right)\left(\left(\prod_{i=n}^{m} A_i\right) + \check{\psi}_m W^*\right) =$$

$$= \left(\left(\prod_{i=n}^{m} A_i\right) U + \left(1_E - \prod_{i=n}^{m} A_i\right)\right) \,.$$

By Proposition 7.1.3 and Proposition 7.1.6 c),

$$a = [U]_1 = \left[\left(\prod_{i=n}^{m} A_i \right) U + \left(1_E - \prod_{i=n}^{m} A_i \right) \right]_1 =$$

$$= \left[\check{\psi}_m \left(\left(\prod_{i=n+1}^{m} A_i \right) V + W \right) \right]_1 =$$

$$= K_1(\psi) \left[\left(\prod_{i=n+1}^{m} A_i \right) V + W \right]_1 \in Im\, K_1(\psi) . \qquad \blacksquare$$

PROPOSITION 7.2.8 *$Ker\, K_0(\varphi) \subset Im\, \delta_1$.*

Let $a \in Ker\, K_0(\varphi)$. By Proposition 6.2.4, there is a $P \in Pr\, \check{F}_{\rightarrow}$ with

$$a = [P]_0 - [\sigma^F_{\rightarrow} P]_0 .$$

By Proposition 6.2.2 c),

$$0 = K_0(\varphi) a = [\check{\varphi}_{\rightarrow} P]_0 - [\check{\varphi}_{\rightarrow} \sigma^F_{\rightarrow} P]_0 .$$

Let $n \in \mathbb{N}$ such that $P \in Pr\, \check{F}_{\rightarrow n}$. Then $[\check{\varphi}_{\rightarrow n} P]_0 = [\check{\varphi}_{\rightarrow n} \sigma^F_{\rightarrow n} P]_0$. By Corollary 6.1.6 a$\Rightarrow$c, there is an $m \in \mathbb{N}$, $m > n+1$, such that

$$\check{\varphi}_{\rightarrow n} P + (B_m)_{\rightarrow} \sim_0 \check{\varphi}_{\rightarrow n} \sigma^F_{\rightarrow n} P + (B_m)_{\rightarrow} .$$

Put

$$Q := P + (B_m)_{\rightarrow} \in Pr\, \check{F}_{\rightarrow m} .$$

Then

$$a = [Q]_0 - [\sigma^F_{\rightarrow} Q]_0, \qquad \check{\varphi}_{\rightarrow m} Q \sim_0 \check{\varphi}_{\rightarrow m} \sigma^F_{\rightarrow m} Q = \sigma^F_{\rightarrow m} Q .$$

By Proposition 6.2.6, there are $k \in \mathbb{N}$, $k \geq m+2$, and $W \in Un\, \check{G}_{\rightarrow k}$ with

$$W(\check{\varphi}_{\rightarrow m} Q) W^* = \sigma^F_{\rightarrow m} Q .$$

It follows

$$(\sigma^F_{\rightarrow m} Q) W = W(\check{\varphi}_{\rightarrow m} Q) W^* W = W(\check{\varphi}_{\rightarrow m} Q) ,$$

$$(\check{\psi}_{\rightarrow k} W)(\sigma^F_{\rightarrow k} Q) = (\check{\psi}_{\rightarrow k} W)(\check{\psi}_{\rightarrow k} \check{\varphi}_{\rightarrow k} Q) = \check{\psi}_{\rightarrow k}(W \check{\varphi}_{\rightarrow k} Q) =$$

$$= \check{\psi}_{\rightarrow k}((\sigma^F_{\rightarrow k}Q)W) = (\sigma^F_{\rightarrow k}Q)(\check{\psi}_{\rightarrow k}W)\ .$$

Put

$$U := (\check{\psi}_{\rightarrow k}W)(1_E - \sigma^F_{\rightarrow k}Q) + \sigma^F_{\rightarrow k}Q \in \check{H}_{\rightarrow k}\ .$$

Then

$$UU^* = U^*U = 1_E, \qquad\qquad U \in Un\,\check{H}_{\rightarrow k}\ .$$

Put

$$V_1 := (A_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q)W + (B_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q \in \check{G}_{k+1}\ .$$

Then

$$V_1^* = (A_{k+1})_{\rightarrow}W^*(1_E - \sigma^F_{\rightarrow k}Q) + (B_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q\,,$$

$$V_1V_1^* = (A_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q) + (B_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q \in Pr\,E_{k+1}\,,$$

$$V_1^*V_1 = (A_{k+1})_{\rightarrow}W^*(1_E - \sigma^F_{\rightarrow k}Q)W + (B_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q =$$

$$= (A_{k+1})_{\rightarrow}(1_E - W^*(\sigma^F_{\rightarrow k}Q)W) + (B_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q\ .$$

Put

$$Z := (1_E - \sigma^F_{\rightarrow k}Q) + ((C_{k+1})_{\rightarrow} + (C^*_{k+1})_{\rightarrow})\sigma^F_{\rightarrow k}Q \in E_{k+1}\ .$$

By Proposition 6.2.5 a),

$$Z^2 = (1_E - \sigma^F_{\rightarrow k}Q) + \sigma^F_{\rightarrow k}Q = 1_E, \qquad\qquad Z \in Un\,E_{k+1}\,,$$

$$ZV_1 = (A_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q)W + (C^*_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q\,,$$

$$V := ZV_1Z = (A_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q)W(1_E - \sigma^F_{\rightarrow k}Q) +$$

$$+ (C^*_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q)W\sigma^F_{\rightarrow k}Q + (A_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q \in \check{G}_{\rightarrow k+1}\,,$$

$$\check{\psi}_{\rightarrow}V = (A_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q)\check{\psi}_{\rightarrow k}W + (A_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q = (A_{k+1})_{\rightarrow}U\,,$$

$$VV^* = ZV_1V_1^*Z \in Pr\,E_{k+1}, \qquad\qquad V^*V = ZV_1^*V_1Z\,,$$

$$1_E - VV^* = Z(1_E - V_1V_1^*)Z =$$

$$= Z((A_{k+1})_{\rightarrow}\sigma^F_{\rightarrow k}Q + (B_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q))Z\,,$$

$$1_E - V^*V = Z(1_E - V_1^*V_1)Z =$$

$$= Z((A_{k+1})_{\rightarrow}W^*(\sigma^F_{\rightarrow k}Q)W + (B_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q))Z =$$

$$= Z((A_{k+1})_{\rightarrow}\check{\varphi}_{\rightarrow k}Q + (B_{k+1})_{\rightarrow}(1_E - \sigma^F_{\rightarrow k}Q))Z\,,$$

$$\check{\varphi}_{\to,k+1}(Z((A_{k+1})_{\to}Q+(B_{k+1})_{\to}(1_E-\sigma^F_{\to k}Q))Z)=$$
$$=Z((A_{k+1})_{\to}\check{\varphi}_k Q+(B_{k+1})_{\to}(1_E-\sigma^F_{\to k}Q))Z=1_E-V^*V\,,$$
$$\check{\varphi}_{\to,k+1}(Z((A_{k+1})_{\to}\sigma^F_{\to k}Q+(B_{k+1})_{\to}(1_E-\sigma^F_{\to k}Q))Z))=1_E-VV^*\,.$$

By Proposition 7.2.6,

$$\delta_1[U]_1=[Z((A_{k+1})_{\to}Q+(B_{k+1})_{\to}(1_E-\sigma^F_{\to k}Q))Z]_0-$$
$$-[Z((A_{k+1})_{\to}\sigma^F_{\to k}Q+(B_{k+1})_{\to}(1_E-\sigma^F_{\to k})Q)Z]_0=[Q]_0-[\sigma^F_{\to}Q]_0=a\,.$$

Thus $a\in Im\;\delta_1$. ■

THEOREM 7.2.9 *The sequence*

$$K_1(F)\xrightarrow{K_1(\varphi)}K_1(G)\xrightarrow{K_1(\psi)}K_1(H)\xrightarrow{\delta_1}K_0(F)\xrightarrow{K_0(\varphi)}K_0(G)\xrightarrow{K_0(\psi)}K_0(H)$$

is exact.

The exactness was proved: for $K_1(G)$ in Proposition 7.1.9, for $K_1(H)$ in Proposition 7.2.7 and Proposition 7.2.5 a), for $K_0(F)$ in Proposition 7.2.8 and Proposition 7.2.5 b), and for $K_0(G)$ in Proposition 6.2.8 c). ■

7.3 $K_1(F)\approx K_0(SF)$

DEFINITION 7.3.1 *Let F be an E-C*-algebra. We denote by CF the E-C*-algebra of continuous maps $x:[0,1]\longrightarrow F$ with $x(0)=0$ and by SF its E-C*-subalgebra* $\{\,x\in CF\mid x(1)=0\}$ (Definition 2.1.1 or [2] Corollary 1.2.5 a),d)). *Moreover we denote by $\theta_F: K_1(F)\longrightarrow K_0(SF)$ the index map associated to the exact sequence*

$$0\longrightarrow SF\xrightarrow{i_F}CF\xrightarrow{j_F}F\longrightarrow 0\,,$$

in $\mathfrak{M}_E$, where i_F is the inclusion map and

$$j_F: CF\longrightarrow F\,,\quad x\longmapsto x(1)\,.$$

If $F\xrightarrow{\varphi}G$ is a morphism in $\mathfrak{M}_E$ then we put

$$S\varphi: SF\longrightarrow SG\,,\quad x\longmapsto \varphi\circ x\,,$$
$$C\varphi: CF\longrightarrow CG\,,\quad x\longmapsto \varphi\circ x\,.$$

If $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ are morphisms in $\mathfrak{M}_E$ then $S(\psi) \circ S(\varphi) = S(\psi \circ \varphi)$.

THEOREM 7.3.2 *θ_F is a group isomorphism for every E-C*-algebra F.*

CF is null-homotopic ([4] Example 4.1.5 or Proposition 2.4.1), so by the Homotopy invariance (Theorem 6.2.11 e), Proposition 7.1.8 e)), it is K-null. By Theorem 7.2.9, the sequence

$$K_1(CF) \xrightarrow{K_1(j_F)} K_1(F) \xrightarrow{\theta_F} K_0(SF) \xrightarrow{K_0(i_F)} K_0(CF)$$

is exact, so θ_F is a group isomorphism. ∎

PROPOSITION 7.3.3 *Let F and G be E-C*-algebras.*

a) For all $(x,y) \in (SF) \times (SG)$ put

$$\overbrace{(x,y)} : [0,1] \longrightarrow F \times G, \quad s \longmapsto (x(s), y(s)) .$$

Then the map

$$(SF) \times (SG) \longrightarrow S(F \times G), \quad (x,y) \longmapsto \overbrace{(x,y)}$$

is an isomorphism in $\mathfrak{M}_E$ (Definition 1.1.2).

b) $K_1(F) \times K_1(G) \approx K_1(F \times G)$ **(Product Theorem)**.

a) is easy to see.

b) By Theorem 7.3.2, the maps

$$K_1(F) \times K_1(G) \xrightarrow{\theta_F \times \theta_G} K_0(SF) \times K_0(SG), \qquad K_1(F \times G) \xrightarrow{\theta_{F\times G}} K_0(S(F \times G))$$

are group isomorphisms. By a), $K_0((SF) \times (SG)) \approx K_0(S(F \times G))$ and by Corollary 6.2.10 b), $K_0((SF) \times (SG)) \approx K_0(SF) \times K_0(SG)$. Thus

$$K_1(F) \times K_1(G) \approx K_1(F \times G) .$$ ∎

COROLLARY 7.3.4 *Let $F \xrightarrow{\varphi} F'$, $G \xrightarrow{\psi} G'$ be morphisms in $\mathfrak{M}_E$ and*

$$\varphi \times \psi : F \times G \longrightarrow F' \times G' , \quad (x,y) \longmapsto (\varphi x, \psi y) .$$

Then $\varphi \times \psi$ is a morphism in $\mathfrak{M}_E$ and

$$K_i(\varphi \times \psi) = K_i(\varphi) \times K_i(\psi)$$

for all $i \in \{0,1\}$.

The assertion follows easily from Corollary 6.2.10 b) and Proposition 7.3.3 b). ■

PROPOSITION 7.3.5 (Product Theorem) *Let $(F_j)_{j\in J}$ be a finite family of E-C*-algebras, $F := \prod_{j\in J} F_j$* (Definition 1.1.2)*, and for every $j \in J$ let $\varphi_j : F_j \longrightarrow F$ be the canonical inclusion and $\psi_j : F \longrightarrow F_j$ the projection. Then for every $i \in \{0,1\}$,*

$$\Phi : \prod_{j\in J} K_i(F_j) \longrightarrow K_i(F) , \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j)a_j$$

is a group isomorphism and

$$\Psi : K_i(F) \longrightarrow \prod_{j\in J} K_i(F_j) , \quad a \longmapsto (K_i(\psi_j)a)_{j\in J}$$

is its inverse.

Φ and Ψ are obviously group homomorphisms. For $j,k \in J$, $\psi_j \circ \varphi_k = 0$ if $j \neq k$ and $\psi_j \circ \varphi_j = id_{F_j}$. Thus for $(a_j)_{j\in J} \in \prod_{j\in J} K_i(F_j)$ and $k \in J$,

$$(\Psi\Phi(a_j)_{j\in J})_k = K_i(\psi_k) \sum_{j\in J} K_i(\varphi_j)a_j = a_k$$

i.e. $\Psi \circ \Phi$ is the identity map of $\prod_{j\in J} K_i(F_j)$. Since $\sum_{j\in J} \varphi_j \circ \psi_j = id_F$, for $a \in K_i(F)$,

$$\Phi\Psi a = \Phi(K_i(\psi_j)a)_{j\in J} = \sum_{j\in J} K_i(\varphi_j)K_i(\psi_j)a = K_i\left(\sum_{j\in J} \varphi_j \circ \psi_j\right) a = a$$

i.e. $\Phi \circ \Psi = id_{K_i(F)}$. ■

THEOREM 7.3.6 (Continuity of K_1) *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in $\mathfrak{M}_E$ and let $\{F, (\varphi_i)_{i\in I}\}$ be its limit in $\mathfrak{M}_E$. By* Proposition 7.1.8 a),

$$\{(K_1(F_i))_{i\in I}, (K_1(\varphi_{ij}))_{i,j\in I}\}$$

is an inductive system in the category of additive groups. Let $\{\mathscr{G}, (\psi_i)_{i\in I}\}$ be its limit in this category and let $\psi : \mathscr{G} \longrightarrow K_1(F)$ be the group homomorphism such that $\psi \circ \psi_i = K_1(\varphi_i)$ for every $i \in I$. Then ψ is a group isomorphism.

By [4] Exercise 10.2, $\{SF, (S\varphi_i)_{i\in I}\}$ is the limit in $\mathfrak{M}_E$ of the inductive system $\{(SF_i)_{i\in I}, (S\varphi_{ij})_{i,j\in I}\}$. By Theorem 6.2.12, $\{K_0(SF), (K_0(S\varphi_i))_{i\in I}\}$ may be identified with the inductive limit in the category of additive groups of the inductive system $\{K_0(SF_i)_{i\in I}, (K_0(S\varphi_{ij}))_{i,j\in I}\}$ and the assertion follows from Theorem 7.3.2. ■

PROPOSITION 7.3.7 *Let F be an E-C*-algebra, $n \in \mathbb{N}$, $U \in Un\,\check{F}_{n-1}$, $V \in Un\,(\overbrace{CF}^{\vee})_n$, and $P \in Pr\,(\overbrace{SF}^{\vee})_n$ such that*

$$\check{j}_F V = A_n U + B_n U^*, \qquad \check{i}_F P = V A_n V^* \;.$$

Then

$$\theta_F[U]_1 = [P]_0 - [\sigma_n^{SF} P]_0.$$

The assertion follows from Corollary 7.2.3 and Definition 7.3.1. ■

PROPOSITION 7.3.8 *If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then the diagram*

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\varphi)} & K_1(G) \\ {\scriptstyle\theta_F}\big\downarrow & & \big\downarrow{\scriptstyle\theta_G} \\ K_0(SF) & \xrightarrow[K_0(S\varphi)]{} & K_0(SG) \end{array}$$

is commutative.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SF & \xrightarrow{i_F} & CF & \xrightarrow{j_F} & F & \longrightarrow & 0 \\ & & {\scriptstyle S\varphi}\big\downarrow & & {\scriptstyle C\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & SG & \xrightarrow[i_G]{} & CG & \xrightarrow[j_G]{} & G & \longrightarrow & 0 \end{array}$$

is commutative and the assertion follows from Proposition 7.2.4. ■

Remark. By Theorem 7.3.2 and Proposition 7.3.8, the functor K_1 is determined by the functor K_0.

COROLLARY 7.3.9 (Split Exact Theorem) *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\overleftarrow{\gamma}}{\overset{\psi}{\rightarrow}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ then

$$0 \longrightarrow K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \underset{\xleftarrow[K_1(\gamma)]{}}{\xrightarrow{K_1(\psi)}} K_1(H) \longrightarrow 0$$

is also split exact. In particular the map

$$K_1(F) \times K_1(H) \longrightarrow K_1(G), \quad (a,b) \longmapsto K_1(\varphi)a + K_1(\lambda)b$$

is a group isomorphism and $K_1\left(\check{F}\right) \approx K_1(E) \times K_1(F)$.

By Theorem 7.2.9, the sequence

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)$$

is exact and by Proposition 7.1.8 a) and Proposition 7.1.6 d),

$$K_1(\psi) \circ K_1(\gamma) = K_1(\psi \circ \gamma) = K_1(id_H) = id_{K_1(H)} \,.$$

It remains only to prove that $K_1(\varphi)$ is injective.

It is easy to see that

$$0 \longrightarrow SF \xrightarrow{S\varphi} SG \underset{\overleftarrow{S\gamma}}{\overset{S\psi}{\rightarrow}} SH \longrightarrow 0$$

is split exact. By Proposition 6.2.9, $K_0(S\varphi)$ is injective and by Proposition 7.3.8, the diagram

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\varphi)} & K_1(G) \\ {\scriptstyle\theta_F}\downarrow & & \downarrow{\scriptstyle\theta_G} \\ K_0(SF) & \xrightarrow[K_0(S\varphi)]{} & K_0(SG) \end{array}$$

is commutative. Since θ_F is injective (Theorem 7.3.2), $K_1(\varphi)$ is also injective.

The last assertion follows from the fact that

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\rightarrow}} E \longrightarrow 0$$

is split exact. ■

COROLLARY 7.3.10 *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\overleftarrow{\gamma}}{\overset{\psi}{\rightarrow}} H \longrightarrow 0\,, \qquad 0 \longrightarrow F' \xrightarrow{\varphi'} G' \underset{\overleftarrow{\gamma'}}{\overset{\psi'}{\rightarrow}} H' \longrightarrow 0$$

be split exact sequences in $\mathfrak{M}_E$ *and*

$$F \xrightarrow{\lambda} F', \qquad G \xrightarrow{\mu} G', \qquad H \xrightarrow{\nu} H'$$

morphisms in $\mathfrak{M}_E$ *such that the corresponding diagram is commutative and let* $i \in \{0,1\}$.

a) If we denote by

$$\phi : K_i(F) \times K_i(H) \longrightarrow K_i(G)\,, \quad (a,b) \longmapsto K_i(\varphi)a + K_i(\gamma)b\,,$$

$$\phi' : K_i(F') \times K_i(H') \longrightarrow K_i(G')\,, \quad (a',b') \longmapsto K_i(\varphi')a' + K_i(\gamma')b'$$

the group isomorphisms (Proposition 6.2.9, Corollary 7.3.9) *then*

$$K_i(\mu) \circ K_i(\phi) = K_i(\phi') \circ (K_i(\lambda) \times K_i(\nu))\ .$$

b) If we identify $K_i(G)$ *with* $K_i(F) \times K_i(H)$ *using* ϕ *and* $K_i(G')$ *with* $K_i(F') \times K_i(H')$ *using* ϕ' *then*

$$K_i(\mu) : K_i(G) \longrightarrow K_i(G')\,, \quad (a,b) \longmapsto (K_i(\lambda)a, K_i(\nu)b)\ .$$

a) For $(a,b) \in K_i(F) \times K_i(H)$,

$$K_i(\mu)K_i(\phi)(a,b) = K_i(\mu)(K_i(\varphi)a + K_i(\gamma)b) =$$

$$= K_i(\varphi')K_i(\lambda)a + K_i(\gamma')K_i(\nu)b = K_i(\phi')(K_i(\lambda) \times K_i(\nu))(a,b)\ .$$

b) follows from a). ■

Chapter 8

Bott Periodicity

8.1 The Bott Map

LEMMA 8.1.1 *Let F be a full E-C*-algebra and $n \in \mathbb{N}$. We identify SF with $\mathscr{C}_0(\mathbb{T} \setminus \{1\}, F)$ in an obvious way.*

a) $F_{\mathbb{T}} := \{ X \in \mathscr{C}(\mathbb{T}, F) \mid X(1) \in E \}$ *is a full E-C*-subalgebra of* $\mathscr{C}(\mathbb{T}, F)$.

b) *If we put for every* $(\alpha, x) \in \overbrace{SF}^{\vee}$

$$\overbrace{(\alpha,x)} : \mathbb{T} \longrightarrow F\,, \quad z \longmapsto \alpha + x(z)$$

then the map

$$\psi : \overbrace{SF}^{\vee} \longrightarrow F_{\mathbb{T}}\,, \quad (\alpha,x) \longmapsto \overbrace{(\alpha,x)}$$

is an E-C-isomorphism. Thus the map*

$$\psi_n : \left(\overbrace{SF}^{\vee} \right)_n \longrightarrow (F_{\mathbb{T}})_n$$

is also an E-C-isomorphism.*

c) *For every* $Y \in (F_{\mathbb{T}})_n$ *put*

$$\ddot{Y} : \mathbb{T} \longrightarrow F_n\,, \quad z \longmapsto \sum_{t \in T_n} (Y_t(z) \otimes id_K) V_t\ .$$

Then $\ddot{Y} \in \{ X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \}$ *for every* $Y \in (F_{\mathbb{T}})_n$ *and the map*

$$\phi^n : (F_{\mathbb{T}})_n \longrightarrow \{ X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \}\,, \quad Y \longmapsto \ddot{Y}$$

is an E-C-isomorphism.*

d) *The map*

$$\phi^n \circ \psi_n : \left(\overbrace{SF}^{\vee} \right)_n \longrightarrow \{ X \in \mathscr{C}(\mathbb{T}, F_n) \mid X(1) \in E_n \}$$

is an E-C-isomorphism. We identify these two full E-C*-algebras by using this isomorphism.The map*

$$Un \left(\overbrace{SF}^{\vee} \right)_n \longrightarrow \{ X \in \mathscr{C}(\mathbb{T}, Un\, F_n) \mid X(1) \in Un\, E_n \}$$

defined by $\phi^n \circ \psi_n$ *is a homeomorphism.*

e) For every

$$X := \sum_{t\in T_n} ((\alpha_t, X_t)\otimes id_K)V_t \in \left(\overset{\vee}{\overbrace{SF}}\right)_n$$

and $z \in \mathbb{T}$*,*

$$(\phi^n \psi_n X)(z) = \sum_{t\in T_n} ((\alpha_t + X_t(z))\otimes id_K)V_t \in F_n\,,$$

$$(\phi^n \psi_n X)(1) = \sum_{t\in T_n} (\alpha_t \otimes id_K)V_t \in E_n\,.$$

f) Consider the split exact sequence in $\mathfrak{M}_E$ (Definition 4.1.4)

$$0 \longrightarrow SF \overset{\iota^{SF}}{\longrightarrow} \overset{\vee}{\overbrace{SF}} \underset{\overset{\longleftarrow}{\lambda^{SF}}}{\overset{\pi^{SF}}{\longrightarrow}} E \longrightarrow 0\,.$$

Then

$$\left(\pi^{SF}\right)_n X = (\phi^n \psi_n X)(1)$$

for every $X \in \left(\overset{\vee}{\overbrace{SF}}\right)_n$.

g) If $F \overset{\varphi}{\to} G$ *is a morphism in* $\mathfrak{C}_E$ *then, by the identification of d), for every* $X \in \mathscr{C}(\mathbb{T}, F_n)$ *with* $X(1) \in E_n$ *and for every* $z \in \mathbb{T}$*,*

$$\left(\left(\overset{\vee}{\overbrace{S\varphi}}\right)_n X\right)(z) = \varphi_n X(z)\,.$$

a) is obvious.

b) For $(\alpha, x), (\beta, y) \in \overset{\vee}{\overbrace{SF}}$, $\gamma \in E$, and $z \in \mathbb{T}$,

$$(\overbrace{(\alpha,x)})^*(z) = \alpha^* + x(z)^* = \overbrace{(\alpha,x)^*}(z)\,,$$

$$(\overbrace{(\alpha,x)}(z))(\overbrace{(\beta,y)}(z)) = (\alpha + x(z))(\beta + y(z)) = \alpha\beta + \alpha y(z) + x(z)\beta + x(z)y(z) =$$

$$= \overbrace{(\alpha\beta, \alpha y + \beta x + xy)}(z) = \overbrace{(\alpha,x)(\beta,y)}(z)\,,$$

$$\overbrace{(\gamma,0)}(z) = \gamma,$$

so ψ is an E-C*-homomorphism. If $\overbrace{(\alpha,x)} = 0$ then for all $z \in \mathbb{T}$

$$\alpha = \alpha + x(1) = 0\,, \qquad x(z) = \alpha + x(z) = 0\,, \qquad x = 0\,,$$

so ψ is injective.

Let $X \in F_{\mathbb{T}}$ and put $\alpha := X(1) \in E$ and

$$x : \mathbb{T} \longrightarrow F, \quad z \longmapsto X(z) - X(1) .$$

Then $(\alpha, x) \in \overbrace{SF}^{\vee}$ and for $z \in \mathbb{T}$,

$$\overbrace{(\alpha, x)}(z) = \alpha + x(z) = X(1) + X(z) - X(1) = X(z) .$$

Thus $\overbrace{(\alpha, x)} = X$ and ψ is surjective.

By [2] Corollary 2.2.5 and [2] Theorem 2.1.9 a), ψ_n is an isomorphism.

c) follows from [2] Proposition 2.3.7 and [2] Theorem 2.1.9 a).

d) follows from b) and c).

e) We have

$$\psi_n X = \sum_{t \in T_n} (\overbrace{(\alpha_t, X_t)} \otimes id_K) V_t ,$$

$$(\phi^n \psi_n X)(z) = \sum_{t \in T_n} ((\alpha_t + X_t(z)) \otimes id_K) V_t \in F_n ,$$

$$(\phi^n \psi_n X)(1) = \sum_{t \in T_n} (\alpha_t \otimes id_K) V_t \in E_n .$$

f) and g) follow from e). ■

DEFINITION 8.1.2 *We put for every full E-C*-algebra F, $n \in \mathbb{N}$, and $P \in F_n$,*

$$\widetilde{P} : \mathbb{T} \longrightarrow F_n , \quad z \longmapsto zP + (1_E - P) .$$

By the identification of Lemma 8.1.1 d),

$$\widetilde{P} \in \{ X \in \mathscr{C}(\mathbb{T}, Un\, F_n) \mid X(1) \in E_n \} = Un \left(\overbrace{SF}^{\vee} \right)_n$$

for every $P \in Pr\, F_n$. Obviously, $\widetilde{0} = 1_E$ and $\widetilde{1_E} = z1_E$.

PROPOSITION 8.1.3 *If F is a full E-C*-algebra, $n \in \mathbb{N}$, and $P \in Pr\, F_{n-1}$ then*

$$\bar{\tau}_n^{\overbrace{SF}^{\vee}} \widetilde{P} = \widetilde{\bar{\rho}_n^F P},$$

(with the identification of Lemma 8.1.1 d)*). Thus we get a well-defined map*

$$\nu_F : Pr\, F_{\rightarrow} \longrightarrow un\, \overbrace{SF}^{\vee}$$

with $\nu_F P = \widetilde{P}$ for every $P \in Pr\, F_{\rightarrow} = \bigcup_{n \in \mathbb{N}} Pr\, F_{\rightarrow n}$.

For $z \in \mathbb{T}$,

$$(\bar{\tau}_n^{\overbrace{SF}^{\vee}} \widetilde{P})(z) = (A_n \widetilde{P} + B_n)(z) = A_n(zP + (1_E - P)) + B_n =$$

$$= zA_n P + (1_E - A_n P) = \widetilde{\bar{\rho}_n^F P}(z)\ . \qquad \blacksquare$$

PROPOSITION 8.1.4 *For every full E-C*-algebra F there is a unique group homomorphism*

$$\beta_F : K_0(F) \longrightarrow K_1(SF) \qquad \textbf{(the Bott map)}$$

such that for every $P \in Pr\, F_{\rightarrow}$,

$$\beta_F [P]_0 = (\nu_F P)/\sim_1 = \left[\tilde{P}\right]_1\ .$$

Let $P, Q \in Pr\, F_{\rightarrow}$ with $P \sim_0 Q$. By Proposition 6.2.6, there are $m, n \in \mathbb{N}$, $m \geq n+2$, and $U \in Un_0\, F_m$ with $P, Q \in Pr\, F_n$ and $UPU^* = Q$ and so

$$(U\widetilde{P}U^*)(z) = U\widetilde{P}(z)U^* = zUPU^* + (1_E - UPU^*) = \widetilde{Q}(z)$$

for every $z \in \mathbb{T}$. Thus $U\widetilde{P}U^* = \widetilde{Q}$, $\widetilde{P} \sim_h \widetilde{Q}$, and $\widetilde{P} \sim_1 \widetilde{Q}$.

Let $P, Q \in Pr\, F_{\rightarrow}$ with $PQ = 0$. We may assume $P, Q \in Pr\, F_{n-1}$ with $P = PA_n$ and $Q = QB_n$ for some $n \in \mathbb{N}$ (Proposition 6.1.3). For every $z \in \mathbb{T}$,

$$\widetilde{P}(z) = zPA_n + (1_E - PA_n), \qquad \widetilde{Q}(z) = zQB_n + (1_E - QB_n)\,,$$

$$(\widetilde{P}\widetilde{Q})(z) = \widetilde{P}(z)\widetilde{Q}(z) = zPA_n + zQB_n + 1_E - QB_n - PA_n =$$

$$= z(P+Q) + (1_E - (P+Q)) = \widetilde{(P+Q)}(z), \qquad \widetilde{P}\widetilde{Q} = \widetilde{P+Q}\,.$$

By Proposition 6.1.9, there is a unique group homomorphism

$$\beta_F : K_0(F) \longrightarrow K_1(SF)$$

with the required property. ■

PROPOSITION 8.1.5 *Let F be an E-C^*-algebra.*

a) There is a unique map $\beta_F : K_0(F) \longrightarrow K_1(SF)$ (called **the Bott map***) such that the diagram*

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\iota^F)} & K_0(\check{F}) \\ {\scriptstyle \beta_F}\downarrow & & \downarrow{\scriptstyle \beta_{\check{F}}} \\ K_1(SF) & \xrightarrow[K_1(S\iota^F)]{} & K_1(S\check{F}) \end{array}$$

is commutative. β_F is a group homomorphism.

b) If F is a full E-C^*-algebra then the above map β_F coincides with the map β_F defined in Proposition 8.1.4.

c) If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then the diagram

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) \\ {\scriptstyle \beta_F}\downarrow & & \downarrow{\scriptstyle \beta_G} \\ K_1(SF) & \xrightarrow[K_1(S\varphi)]{} & K_1(SG) \end{array}$$

is commutative.

c) for $\mathfrak{C}_E$ with $F \xrightarrow{\varphi} G$ unital. For $n \in \mathbb{N}$, $P \in Pr\, F_n$, and $z \in \mathbb{T}$, by Lemma 8.1.1 g),

$$\left(\left(\overset{\vee}{\widehat{S\varphi}}\right)_n \widetilde{P}\right)(z) = z\varphi_n P + (1_E - \varphi_n P) = \left(\widetilde{\varphi_n P}\right)(z)\,,$$

$$\left(\overset{\vee}{\widehat{S\varphi}}\right)_n \widetilde{P} = \widetilde{\varphi_n P}\,.$$

By Proposition 6.1.10 c), Proposition 8.1.4, and Proposition 7.1.6 c),

$$K_1(S\varphi)\beta_F[P]_0 = K_1(S\varphi)\left[\widetilde{P}\right]_1 =$$

$$= \left[\left(\overbrace{S\varphi}^{\vee}\right)_n \widetilde{P}\right]_1 = \left[\widetilde{\varphi_n P}\right]_1 = \beta_G[\varphi_n P]_0 = \beta_G K_0(\varphi)[P]_0\,,$$

$$K_1(S\varphi)\circ\beta_F = \beta_G\circ K_0(\varphi)\,.$$

a) By c) for $\mathfrak{C}_E$, the diagram

$$\begin{array}{ccc} K_0(\check{F}) & \xrightarrow{K_0(\pi^F)} & K_0(E) \\ \beta_{\check{F}}\big\downarrow & & \big\downarrow\beta_E \\ K_1(S\check{F}) & \xrightarrow[K_1(S\pi^F)]{} & K_1(SE) \end{array}$$

is commutative. By Proposition 6.1.12 c) and Corollary 7.3.9 the sequences

$$0 \longrightarrow K_0(F) \xrightarrow{K_0(\iota^F)} K_0(\check{F}) \xrightarrow{K_0(\pi^F)} K_0(E) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(SF) \xrightarrow{K_1(S\iota^F)} K_1(S\check{F}) \xrightarrow{K_1(S\pi^F)} K_1(SE) \longrightarrow 0$$

are exact, since the sequence

$$0 \longrightarrow SF \xrightarrow{S\iota^F} S\check{F} \underset{\xleftarrow[S\lambda^F]{}}{\xrightarrow{S\pi^F}} SE \longrightarrow 0$$

is split exact. By the above c) for $\mathfrak{C}_E$, Corollary 6.2.3 a), and Proposition 6.2.2 e),

$$K_1(S\pi^F)\circ\beta_{\check{F}}\circ K_0(\iota^F) = \beta_E\circ K_0(\pi^F)\circ K_0(\iota^F) =$$

$$= \beta_E\circ K_0(\pi^F\circ\iota^F) = \beta_E\circ K_0(0) = 0\,.$$

Thus

$$Im\,(\beta_{\check{F}}\circ K_0(\iota^F)) \subset Ker\,K_1(S\pi^F) = Im\,K_1(S\iota^F)\,.$$

The assertion follows now from the fact that $K_1(S\iota^F)$ is injective.

b) By c) for $\mathfrak{C}_E$, the diagram

$$\begin{array}{ccc} K_0(F) & \xrightarrow{K_0(\iota^F)} & K_0(\check{F}) \\ \beta_F\big\downarrow & & \big\downarrow\beta_{\check{F}} \\ K_1(SF) & \xrightarrow[K_1(S\iota^F)]{} & K_1(S\check{F}) \end{array}$$

is commutative, with β_F defined in Proposition 8.1.4. By a), this β_F coincides with β_F defined in a).

c) The following diagrams

$$
\begin{array}{ccc}
F & \xrightarrow{\varphi} & G \\
{\scriptstyle \iota^F}\downarrow & & \downarrow{\scriptstyle \iota^G} \\
\check{F} & \xrightarrow[\check{\varphi}]{} & \check{G}
\end{array}
\qquad
\begin{array}{ccc}
SF & \xrightarrow{S\varphi} & SG \\
{\scriptstyle S\iota^F}\downarrow & & \downarrow{\scriptstyle S\iota^G} \\
S\check{F} & \xrightarrow[S\check{\varphi}]{} & S\check{G}
\end{array}
\qquad
\begin{array}{ccc}
K_1(SF) & \xrightarrow{K_1(S\varphi)} & K_1(SG) \\
{\scriptstyle K_1(S\iota^F)}\downarrow & & \downarrow{\scriptstyle K_1(S\iota^G)} \\
K_1(S\check{F}) & \xrightarrow[K_1(S\check{\varphi})]{} & K_1(S\hat{G})
\end{array}
$$

are obviously commutative (Proposition 7.1.8 a)). So by a) and c) for $\mathfrak{C}_E$ (and Corollary 6.2.3 a), Proposition 7.1.8 a)),

$$K_1(S\iota^G)\circ\beta_G\circ K_0(\varphi)=\beta_{\check{G}}\circ K_0(\iota^G)\circ K_0(\varphi)=\beta_{\check{G}}\circ K_0(\check{\varphi})\circ K_0(\iota^F)=$$

$$=K_1(S\check{\varphi})\circ\beta_{\check{F}}\circ K_0(\iota^F)=K_1(S\check{\varphi})\circ K_1(S\iota^F)\circ\beta_F=K_1(S\iota^G)\circ K_1(S\varphi)\circ\beta_F\ .$$

The assertion follows now from the fact that $K_1(S\iota^G)$ is injective. ■

8.2 Higman's Linearization Trick

Throughout this section F denotes a full E-C*-algebra, $m,n\in\mathbb{N}$, and $l:=2^m-1$.

DEFINITION 8.2.1 *We shall use the following notation* ([4] 11.2)*:*

$$Trig(n):=\left\{X\in\mathscr{C}(\mathbb{T},GL_{E_n}(F_n))\mid X(z)=\sum_{p=-m}^{m}a_pz^p,\,a_p\in F_n\right\},$$

$$Pol(n,m):=\left\{X\in\mathscr{C}(\mathbb{T},GL_{E_n}(F_n))\mid X(z)=\sum_{p=0}^{m}a_pz^p,\,a_p\in F_n\right\},$$

$$Pol(n):=\bigcup_{m\in\mathbb{N}}Pol(n,m),\qquad Lin(n):=Pol(n,1),$$

$$Proj(n):=\left\{\widetilde{P}\;\middle|\;P\in Pr\,F_n\right\}.$$

LEMMA 8.2.2

a) If $X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ then there are $k \in \mathbb{N}$ and $Y \in Pol(n)$ such that $z^k X$ is homotopic to Y in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$.

b) If $P, Q \in Pr\, F_n$ such that $\widetilde{P}$ and $\widetilde{Q}$ are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ then there are $k, m \in \mathbb{N}$ such that $z^k \widetilde{P}$ is homotopic to $z^k \widetilde{Q}$ in $Pol(n, l)$.

a) It is possible to adapt [4] Lemma 11.2.3 to the present situation in order to find a $Z \in Trig(n)$ such that

$$\|X - Z\| < \left\|X^{-1}\right\|^{-1} .$$

By [4] Proposition 2.1.11, X and Z are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$. There is a $k \in \mathbb{N}$ such that $Y := z^k Z \in Pol(n)$. Then $z^k X$ and Y are homotopic in $\mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$.

b) The proof of [4] Lemma 11.2.4 (ii) works in this case too. ■

DEFINITION 8.2.3 *The map*

$$\{0,1\}^m \longrightarrow \mathbb{N}_l \cup \{0\}, \quad j \longmapsto \sum_{i=1}^{m} j_i 2^{i-1}$$

is bijective. We denote by

$$\mathbb{N}_l \cup \{0\} \longrightarrow \{0,1\}^m, \quad p \longmapsto |p|$$

its inverse. For every $i \in \mathbb{N}_m$ and $p, q \in \mathbb{N}_l \cup \{0\}$ we put

$$(p,q)_i := \begin{cases} A_{n+i} & \text{if} \quad |p|_i = |q|_i = 0 \\ C^*_{n+i} & \text{if} \quad |p|_i = 0,\, |q|_i = 1 \\ C_{n+i} & \text{if} \quad |p|_i = 1,\, |q|_i = 0 \\ B_{n+i} & \text{if} \quad |p|_i = |q|_i = 1 \end{cases} .$$

LEMMA 8.2.4

a) For $p, q, r, s \in \mathbb{N}_l \cup \{0\}$ and $i \in \mathbb{N}_m$,

$$(p,q)_i (r,s)_i = \begin{cases} 0 & \text{if} \quad |q|_i \neq |r|_i \\ (p,s)_i & \text{if} \quad |q|_i = |r|_i \end{cases} .$$

In particular

$$\prod_{i=1}^{m}((p,q)_i(r,s)_i)=\begin{cases} 0 & \text{if} \quad q\neq r \\ \prod\limits_{i=1}^{m}(p,s)_i & \text{if} \quad q=r \end{cases}.$$

b) For $p,q\in\mathbb{N}_l\cup\{0\}$ and $i\in\mathbb{N}_m$,

$$A_{n+i}(p,q)_i=\begin{cases} (p,q)_i & \text{if} \quad |p|_i=0 \\ 0 & \text{if} \quad |p|_i=1 \end{cases},$$

$$(p,q)_iA_{n+i}=\begin{cases} (p,q)_i & \text{if} \quad |q|_i=0 \\ 0 & \text{if} \quad |q|_i=1 \end{cases}.$$

In particular

$$p\neq 0\Longrightarrow\prod_{i=1}^{m}(A_{n+i}(p,q)_i)=0\,,$$

$$q\neq 0\Longrightarrow\prod_{i=1}^{m}((p,q)_iA_{n+i})=0\,,$$

$$\sum_{r=q}^{l}\prod_{i=1}^{m}(A_{n+i}(r,r-q)_i)=\begin{cases} 0 & \text{if} \quad q\neq 0 \\ \prod\limits_{i=1}^{m}A_{n+i} & \text{if} \quad q=0 \end{cases}.$$

c) $\sum\limits_{p=0}^{l}\prod\limits_{i=1}^{m}(p,p)_i=1_E$.

a) and b) is a long verification.

c) For every $p\in\mathbb{N}_l\cup\{0\}$ put

$$J_p:=\{\,i\in\mathbb{N}_m\mid |p|_i=0\,\}\,,\qquad K_p:=\{\,i\in\mathbb{N}_m\mid |p|_i=1\,\}\;.$$

Then

$$1_E=\prod_{i=1}^{m}(A_{n+i}+B_{n+i})=\sum_{p=0}^{l}\left(\prod_{i\in J_p}A_{n+i}\right)\left(\prod_{i\in K_p}B_{n+i}\right)=\sum_{p=0}^{l}\prod_{i=1}^{m}(p,p)_i\,. \qquad \blacksquare$$

LEMMA 8.2.5 *Let $a\in(F_n)^l$ and*

$$X:=\sum_{p=1}^{l}a_p\sum_{q=p}^{l}\prod_{i=1}^{m}(q,q-p)_i \qquad (X\in F_{m+n})\;.$$

a) $X^{2^m} = 0.$

b) $1_E - X$ *is invertible.*

a) We put $D := \mathbb{N}_1$ and for every $k \in \mathbb{N}$ and $p \in D^k$,

$$p^{(k)} := \sum_{j=1}^{k} p_j, \qquad a_p^{(k)} := \prod_{j=1}^{k} a_{p_j} .$$

We want to prove by induction that for every $k \in \mathbb{N}$,

$$X^k = \sum_{p \in D^k} a_p^{(k)} \sum_{q=p^{(k)}}^{l} \prod_{i=1}^{m} (q, q - p^{(k)})_i .$$

The assertion holds for $k = 1$. Assume the assertion holds for $k \in \mathbb{N}$. Then

$$X^{k+1} = \sum_{p \in D^k} \sum_{p' \in D} a_p^{(k)} a_{p'} \sum_{q=p^{(k)}}^{l} \sum_{q'=p'}^{l} \prod_{i=1}^{m} ((q, q - p^{(k)})_i (q', q' - p')_i) .$$

By Lemma 8.2.4 a),

$$X^{k+1} = \sum_{p \in D^k} \sum_{p' \in D} a_p^{(k)} a_{p'} \sum_{q=p^{(k)}+p'}^{l} \prod_{i=1}^{m} (q, q - p^{(k)} - p')_i =$$

$$= \sum_{p \in D^{k+1}} a_p^{(k+1)} \sum_{q=p^{(k+1)}}^{l} \prod_{i=1}^{m} (q, q - p^{(k+1)})_i ,$$

which finishes the inductive proof. Since $p^{(k)} \geq k$ for every $k \in \mathbb{N}$ we get $X^{2^m} = 0$.

b) By a), $1_E + \sum_{k=1}^{l} X^k$ is the inverse of $1_E - X$. ■

PROPOSITION 8.2.6 (Higman's linearization trick) *There is a continuous map*

$$\mu : Pol(n,l) \longrightarrow Lin(n+m)$$

such that μX *is homotopic to* $X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$ *in* $Pol(n+m, 2l+1)$ *for every* $X \in Pol(n,l)$. *If* $X \in Proj(n)$ *then the above homotopy takes place in* $Lin(n+1)$.

Assume $X \in Pol(n,l)$ is given by

$$X = \sum_{p=0}^{l} a_p z^p\,,$$

where $a_p \in F_n$ for every $p \in \mathbb{N}_l \cup \{0\}$. Put

$$X_p := \sum_{q=p}^{l} a_q z^{q-p} \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_n))$$

for all $p \in \mathbb{N}_l \cup \{0\}$ and for all $s \in [0,1]$,

$$Y_s := 1_E - s\sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,,$$

$$Z_s := 1_E + s\sum_{q=1}^{l} z^q \sum_{r=q}^{l} \prod_{i=1}^{m} (r, r-q)_i \qquad\qquad (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,.$$

By Lemma 8.2.4 a),

$$Y_s(1_E + s\sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i) = (1_E + s\sum_{p=1}^{l} X_p \prod_{i=1}^{m} (0,p)_i) Y_s =$$

$$= 1_E + s^2 \sum_{p,q=1}^{l} X_p X_q \prod_{i=1}^{m} ((0,p)_i (0,q)_i) = 1_E\,,$$

so Y_s is invertible. By Lemma 8.2.5 b), Z_s is also invertible. Thus for every $s \in [0,1]$, Y_s and Z_s are homotopic to 1_E in $\mathscr{C}(\mathbb{T}, GL(F_{n+m}))$ and belong therefore to $Pol(n+m,l)$. By Lemma 8.2.4 c),

$$Z_1 = \sum_{q=0}^{l} z^q \sum_{r=q}^{l} \prod_{i=1}^{m} (r, r-q)_i\,.$$

Put

$$\mu X := 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} a_p \prod_{i=1}^{m} (0,p)_i - z\sum_{p=1}^{l} \prod_{i=1}^{m} (p, p-1)_i \;\; (\in \mathscr{C}(\mathbb{T}, F_{n+m}))\,.$$

For $z \in \mathbb{T}$,

$$((\mu X)Z_1)(z) = \sum_{p=0}^{l} z^p \sum_{q=p}^{l} \prod_{i=1}^{m} (q, q-p)_i - \sum_{p=0}^{l} z^p \sum_{q=p}^{l} \prod_{i=1}^{m} (A_{n+i}(q, q-p)_i) +$$

$$+\sum_{p,q=0}^{l} a_p z^q \sum_{r=q}^{l}\prod_{i=1}^{m}((0,p)_i(r,r-q)_i) - \sum_{q=0}^{l} z^{q+1}\sum_{p=1}^{l}\sum_{r=q}^{l}\prod_{i=1}^{m}((p,p-1)_i(r,r-q)_i)\,.$$

By Lemma 8.2.4 b),

$$\sum_{p=0}^{l} z^p \sum_{q=p}^{l}\prod_{i=1}^{m}(A_{n+i}(q,q-p)_i) = \prod_{i=1}^{m} A_{n+i}$$

and by Lemma 8.2.4 a),

$$\sum_{p,q=0}^{l} a_p z^q \sum_{r=q}^{l}\prod_{i=1}^{m}((0,p)_i(r,r-q)_i) = \sum_{q=0}^{l} z^q \sum_{p=q}^{l} a_p \prod_{i=1}^{m}(0,p-q)_i =$$

$$= \sum_{q=0}^{l} z^q \sum_{r=0}^{l-q} a_{q+r}\prod_{i=1}^{m}(0,r)_i = \sum_{r=0}^{l}\sum_{q=0}^{l-r} z^q a_{q+r}\prod_{i=1}^{m}(0,r)_i =$$

$$= \sum_{r=0}^{l}\sum_{s=r}^{l} z^{s-r} a_s \prod_{i=1}^{m}(0,r)_i = \sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i\,,$$

$$\sum_{q=0}^{l} z^{q+1}\sum_{p=1}^{l}\sum_{r=q}^{l}\prod_{i=1}^{m}((p,p-1)_i(r,r-q)_i) =$$

$$= \sum_{q=0}^{l} z^{q+1}\sum_{p=q+1}^{l}\prod_{i=1}^{m}(p,p-q-1)_i = \sum_{q=1}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i\,.$$

Thus by Lemma 8.2.4 c),

$$((\mu X)Z_1)(z) = \sum_{q=0}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i - \prod_{i=1}^{m} A_{n+i} +$$

$$+ \sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i - \sum_{q=1}^{l} z^q \sum_{p=q}^{l}\prod_{i=1}^{m}(p,p-q)_i =$$

$$= \sum_{p=0}^{l}\prod_{i=1}^{m}(p,p)_i - \prod_{i=1}^{m} A_{n+i} + \sum_{r=0}^{l} X_r \prod_{i=1}^{m}(0,r)_i = 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} X_p \prod_{i=1}^{m}(0,p)_i\,.$$

By Lemma 8.2.4 a),b), for $z \in \mathbb{T}$,

$$(Y_1(\mu X)Z_1)(z) = 1_E - \prod_{i=1}^{m} A_{n+i} + \sum_{p=0}^{l} X_p \prod_{i=1}^{m}(0,p)_i - \sum_{p=1}^{l} X_p \prod_{i=1}^{m}(0,p)_i +$$

$$+ \sum_{p=1}^{l} X_p \prod_{i=1}^{m}((0,p)_i A_{n+i}) - \sum_{p=1}^{l}\sum_{q=0}^{l} X_p X_q \prod_{i=1}^{m}((0,p)_i(0,q)_i) =$$

$$= 1_E - \prod_{i=1}^{m} A_{n+i} + X_0 \prod_{i=1}^{m} (0,0)_i = 1_E - \prod_{i=1}^{m} A_{n+i} + X \prod_{i=1}^{m} A_{n+i} \,.$$

Since $1_E - \prod_{i=1}^{m} A_{n+i} + X^{-1} \prod_{i=1}^{m} A_{n+i}$ is the inverse of $Y_1(\mu X)Z_1$ it follows that $Y_1(\mu X)Z_1$ and μX are invertible, i.e. they belong to $\mathscr{C}(\mathbb{T}, GL(F_{n+m}))$. Thus for every $s \in [0,1]$, $Y_s(\mu X)Z_s \in \mathscr{C}(\mathbb{T}, GL(F_{n+m}))$. Let $z \in \mathbb{T}$ and let

$$[0,1] \longrightarrow GL(F_n), \quad s \longmapsto x_s$$

be a continuous map with $x_0 = X(z)$ and $x_1 = 1_E$. Since $1_E - \prod_{i=1}^{m} A_{n+i} + x_s^{-1} \prod_{i=1}^{m} A_{n+i}$ is the inverse of $1_E - \prod_{i=1}^{m} A_{n+i} + x_s \prod_{i=1}^{m} A_{n+i}$ for every $s \in [0,1]$ it follows that the map

$$[0,1] \longrightarrow GL(F_{n+m}), \quad s \longmapsto 1_E - \prod_{i=1}^{m} A_{n+i} + x_s \prod_{i=1}^{m} A_{n+i}$$

is well-defined and it is a homotopy from $(Y_1(\mu X)Z_1)(z)$ to 1_E i.e. $Y_1(\mu X)Z_1 \in \mathscr{C}(\mathbb{T}, GL_0(F_{n+m}))$ and $Y_1(\mu X)Z_1 \in Pol(n+m, l)$. By the above, for every $s \in [0,1]$, $Y_s(\mu X)Z_s \in \mathscr{C}(\mathbb{T}, GL_0(F_{n+m}))$, so $Y_s(\mu X)Z_s \in Pol(n+m, 2l+1)$. Hence μX is homotopic to $X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$ in $Pol(n+m, 2l+1)$ and $\mu X \in Lin(n+m)$.

In order to prove the last assertion remark that there is a $P \in Pr\, F_n$ with $X = \widetilde{P} = (1_E - P) + zP$. Then $m = l = 1$, $a_0 = 1_E - P$, $a_1 = P$, $X_1 = a_0 = P$,

$$\mu X = 1_E - PA_{n+1} + PC^*_{n+1} - zC_{n+1} \,,$$

and for every $s \in [0,1]$,

$$Y_s = 1_E - sPC^*_{n+1} \,, \qquad Z_s := 1_E + szC_{n+1} \,, \qquad Y_s(\mu X)Z_s \in Lin(n+1) \,.$$

Thus μX is homotopic to $Y_1(\mu X)Z_1$ in $Lin(n+1)$. ■

8.3 The Periodicity

Throughout this section F denotes a full E-C*-algebra, $m, n \in \mathbb{N}$, and $l := 2^m - 1$.

LEMMA 8.3.1 *If $X \in \mathscr{C}(\mathbb{T}, GL(F_n))$ and $X(1) \in GL_{E_n}(F_n)$ then*

$$X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n)) \,.$$

Let $\theta \in [0, 2\pi[$ and for every $s \in [0,1]$ put

$$Y_s : \mathbb{T} \longrightarrow GL(F_n)\,, \quad z \longmapsto X(e^{-is}z)\,.$$

Then $Y_0(e^{i\theta}) = X(e^{i\theta})$ and $Y_\theta(e^{i\theta}) = X(1)$ so $X(e^{i\theta})$ is homotopic to $X(1)$ in $GL(F_n)$. Thus $X(e^{i\theta}) \in GL_{E_n}(F_n)$ and $X \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$. ■

PROPOSITION 8.3.2 *The following are equivalent for every $X \in F_n$.*

a) $\widetilde{X} \in Lin(n)$.

b) $z \in \mathbb{T} \setminus \{1\} \Longrightarrow \widetilde{X}(z) \in GL(F_n)$.

c) $\widetilde{X}$ *is a generalized idempotent of* F_n ([4] Definition 11.2.8).

$a \Rightarrow b$ is trivial.

$b \Rightarrow a$. By Lemma 8.3.1, since $\widetilde{X}(1) = 1_E$, $\tilde{X} \in \mathscr{C}(\mathbb{T}, GL_{E_n}(F_n))$ so $\tilde{X} \in Lin(n)$.

$b \Leftrightarrow c$. For $z \in \mathbb{T} \setminus \{1\}$,

$$\widetilde{X}(z) = (z-1)X + 1_E = (z-1)\left(X - \frac{1}{1-z}1_E\right)\,.$$

Since

$$\left\{ \frac{1}{1-z} \;\middle|\; z \in \mathbb{T} \setminus \{1\} \right\} = \left\{ \alpha \in \mathbb{C} \mid real(\alpha) = \frac{1}{2} \right\},$$

b) holds iff $X - \alpha 1_E$ is invertible for every $\alpha \in \mathbb{C}$ with $real(\alpha) = \frac{1}{2}$, which is equivalent to c). ■

LEMMA 8.3.3 *For $z \in \mathbb{T}$,*

$$zA_n + B_n \sim_h A_n + zB_n \quad \text{in} \quad Un\, E_n\,.$$

We have

$$(C_n+C_n^*)(zA_n+B_n)(C_n+C_n^*)=(zC_n+C_n^*)(C_n+C_n^*)=zB_n+A_n$$

and the assertion follows from Proposition 6.2.5 a). ■

LEMMA 8.3.4 *For $z\in\mathbb{T}$,*

$$z^l\prod_{i=1}^{m}A_{n+i}+\sum_{p=1}^{l}\prod_{i=1}^{m}(p,p)_i\sim_h\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{l}\prod_{i=1}^{m}(p,p)_i\quad\text{in}\quad Un\,E_{n+m}\,.$$

Let $k\in\mathbb{N}_l$ and let $j\in\mathbb{N}_m$ with $|k|_j=1$. By Lemma 8.3.3,

$$\begin{aligned}
&z^{l-k+1}\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k}^{l}\prod_{i=1}^{m}(p,p)_i=\\
&=\left(z^{l-k}\prod_{i=1}^{m}A_{n+i}+\prod_{i=1}^{m}(k,k)_i\right)(zA_{n+j}+(k,k)_j)+\\
&\quad+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i\sim_h\\
&\sim_h\left(z^{l-k}\prod_{i=1}^{m}A_{n+i}+\prod_{i=1}^{m}(k,k)_i\right)(A_{n+j}+z(k,k)_j)+\\
&\quad+z\sum_{p=1}^{k-1}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i=\\
&=z^{l-k}\prod_{i=1}^{m}A_{n+i}+z\sum_{p=1}^{k}\prod_{i=1}^{m}(p,p)_i+\sum_{p=k+1}^{l}\prod_{i=1}^{m}(p,p)_i
\end{aligned}$$

in $Un\,E_{n+m}$. The assertion follows now by induction on $k\in\mathbb{N}_l$. ■

LEMMA 8.3.5 *Let $P,Q\in Pr\,F_n$.*

a) For every $z\in\mathbb{T}$,

$$\overbrace{P\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}(z)=$$

$$= \widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

b) If (with the identification of Lemma 8.1.1 d)*)*

$$\widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Un\left(\overset{\vee}{\overbrace{SF}}\right)_{n+m},$$

then

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \quad \text{in} \quad Un\left(\overset{\vee}{\overbrace{SF}}\right)_{n+m} .$$

a) We have

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) =$$

$$= zP\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) + \prod_{i=1}^{m} A_{n+i} +$$

$$+ \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) - P\left(\prod_{i=1}^{m} A_{n+i}\right) - \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

b) Let

$$[0,1] \longrightarrow Un\left(\overset{\vee}{\overbrace{SF}}\right)_{n+m}, \quad s \longmapsto U_s$$

be a continuous map with

$$U_0 = \widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) ,$$

$$U_1 = \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

Put $U'_s := U_s\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$ for every $s \in [0,1]$. Then $s \mapsto U'_s$ is a continuous path in $Un\left(\overbrace{SF}^{\vee}\right)_{n+m}$ and by a),

$$U'_0 = U_0\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right) =$$

$$= \overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) ,$$

$$U'_1 = \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) .$$

■

PROPOSITION 8.3.6

a) *If* $U \in Un\left(\overbrace{SF}^{\vee}\right)_n$ *then there are* $k, m \in \mathbb{N}$ *and* $P \in Pr\, F_{n+m}$ *such that (with the identification of* Lemma 8.1.1 d)*)*

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m} .$$

b) *Let* $P, Q \in Pr\, F_n$ *with* $\widetilde{P} \sim_h \widetilde{Q}$ *in* $Un\left(\overbrace{SF}^{\vee}\right)_n$*. Then there is an* $m \in \mathbb{N}$ *such that*

$$P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Pr\, F_{n+m} .$$

a) By Proposition 8.2.2 a), there are $k,m \in \mathbb{N}$, $k < 2^m$, and $X \in Pol(n,l)$ such that $z^k U$ is homotopic to X in $\mathscr{C}(\mathbb{T}, GL_E(F_n))$. By Proposition 8.2.6, there is a $Y \in Lin(n+m)$ with

$$X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h Y \quad \text{in} \quad Pol(n+m, 2l+1)\,.$$

By [4] Lemma 11.2.12 (i), there is a $P \in Pr\, F_{n+m}$ with $Y \sim_h \widetilde{P}$ in $Lin(n+m)$. Thus

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h$$

$$\sim_h X\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h Y \sim_h \widetilde{P}$$

in $\mathscr{C}(\mathbb{T}, GL_E(F_{n+m}))$. By [4] Proposition 2.1.8 (iii) and the identification of Lemma 8.1.1 d),

$$(z^k U)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{n+m}\,.$$

b) By Proposition 8.2.2 b), there are $k,m \in \mathbb{N}$, $k < 2^m$, such that $z^k \widetilde{P} \sim_h z^k \widetilde{Q}$ in $Pol(n,l)$. By Lemma 8.3.4 and Lemma 8.2.4 c),

$$z^l\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \sim_h \left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)$$

in $Un\, E_{n+m}$. By Lemma 8.3.5 a),

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim}(z) =$$

$$= \left(\left(\prod_{i=1}^{m} A_{n+i}\right) + z\left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) \times$$

$$\times \left(\widetilde{P}(z)\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) \sim_h$$

$$\sim_h \left(z^l\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right)\left(\widetilde{P}(z) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right) =$$

$$= z^l\widetilde{P}(z)\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)\sim_h$$

$$\sim_h z^l\widetilde{Q}(z)\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)\sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}(z)$$

in $Pol(n+m,l)$. By Proposition 8.2.6,

$$\widetilde{P}\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)=$$

$$=\overbrace{P\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)\sim_h$$

$$\sim_h \mu\left(\overbrace{P\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}\right)\sim_h$$

$$\sim_h \mu\left(\overbrace{Q\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}\right)\sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)$$

in $Lin(n+m)$. By Lemma 8.3.5 a),

$$\overbrace{P\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}=\widetilde{P}\left(\prod_{i=1}^{m}A_{n+i}\right)+z\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)\sim_h$$

$$\sim_h \widetilde{Q}\left(\prod_{i=1}^{m}A_{n+i}\right)+z\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)=\overbrace{Q\left(\prod_{i=1}^{m}A_{n+i}\right)+\left(1_E-\prod_{i=1}^{m}A_{n+i}\right)}^{\sim}$$

in $Lin(n+m)$. The assertion follows now from [4] Lemma 11.2.12 (ii). ■

THEOREM 8.3.7 *The Bott map is bijective.*

Step 1 Surjectivity

Let $a \in K_1(SF)$. There are $n \in \mathbb{N}$ and $U \in Un\left(\overbrace{SF}^{\vee}\right)_n$ with $a = [U]_1$. By Proposition 8.3.6 a), there are $m, p \in \mathbb{N}$, $p \geq n$, and $P \in Pr\, F_{p+m}$ such that

$$(z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right) \sim_h \widetilde{P} \quad \text{in} \quad Un\left(\overbrace{SF}^{\vee}\right)_{p+m}.$$

By Lemma 8.3.4 and Lemma 8.2.4 c),

$$\overbrace{1_E - \prod_{i=1}^{m} A_{p+i}}^{\sim} = z\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + \left(\prod_{i=1}^{m} A_{p+i}\right) \sim_h$$

$$\sim_h \left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right) \quad \text{in} \quad Un\, E_{p+m}$$

so by Proposition 7.1.3 and Proposition 8.1.4,

$$\beta_F\left([P]_0 - \left[1_E - \prod_{i=1}^{m} A_{p+i}\right]_0\right) = [\widetilde{P}]_1 - \left[\overbrace{1_E - \prod_{i=1}^{m} A_{p+i}}^{\sim}\right]_1 =$$

$$= \left[(z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right]_1 -$$

$$- \left[\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right)\right]_1 =$$

$$= \left[\left((z^l U)\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right) \times\right.$$

$$\left.\times \left(\left(1_E - \prod_{i=1}^{m} A_{p+i}\right) + z^l\left(\prod_{i=1}^{m} A_{p+i}\right)\right)^*\right]_1 =$$

$$= \left[U\left(\prod_{i=1}^{m} A_{p+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{p+i}\right)\right]_1 = [U]_1 = a\,.$$

Step 2 Injectivity

Let $a \in K_0(F)$ with $\beta_F a = 0$. By Proposition 6.1.5 d), there are $P, Q \in Pr\, F_n$, $PQ = 0$, such that $a = [P]_0 - [Q]_0$. Then $[\widetilde{P}]_1 = [\widetilde{Q}]_1$, so $U := \tilde{P}\tilde{Q}^* \in un_{E_n} \overbrace{SF}^{\vee}$. Then

$$U = ((z-1)P + 1_E)((\bar{z}-1)Q + 1_E)) = (z-1)P + (\bar{z}-1)Q + 1_E\,, \qquad U(1) = 1_E\,.$$

By Proposition 7.1.3, there is an $m \in \mathbb{N}$ such that

$$V := U\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = \tau^F_{n+m,n} U \in Un_{E_{n+m}} \left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

Then there is a $W \in Un\, E_{n+m}$ with $V \sim_h W$ in $Un \left(\overbrace{SF}^{\vee}\right)_{n+m}$. By the above,

$$W = W(1) \sim_h V(1) = 1_E\,, \qquad V \sim_h 1_E \quad \text{in} \quad Un \left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

By Proposition 7.1.3,

$$\widetilde{P}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = \tau^F_{n+m,n}\tilde{P} = (\tau^F_{n+m,n}U)(\tau^F_{n+m,n}\tilde{Q}) =$$

$$= V(\tau^F_{n+m,n}\tilde{Q}) \sim_h \widetilde{Q}\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \quad \text{in} \quad Un \left(\overbrace{SF}^{\vee}\right)_{n+m},$$

so by Proposition 8.3.5 b),

$$\overbrace{P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \sim_h$$

$$\sim_h \overbrace{Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)}^{\sim} \quad \text{in} \quad Un \left(\overbrace{SF}^{\vee}\right)_{n+m}.$$

Put

$$P' := P\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right),$$

$$Q' := Q\left(\prod_{i=1}^{m} A_{n+i}\right) + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) .$$

By Proposition 8.3.6 b), there are $m', p' \in \mathbb{N}$ such that

$$P'\left(\prod_{j=1}^{m'} A_{p'+j}\right) + \left(1_E - \prod_{j=1}^{m'} A_{p'+i}\right) \sim_h$$

$$\sim_h \; Q'\left(\prod_{j=1}^{m'} A_{p'+i}\right) + \left(1_E - \prod_{j=1}^{m'} A_{p'+i}\right) \quad \text{in} \quad Pr\, F_{p'+m'} .$$

It follows successively

$$\left[P'\prod_{j=1}^{m'} A_{p'+j}\right]_0 = \left[Q'\prod_{j=1}^{m'} A_{p'+j}\right]_0 ,$$

$$\left[P\left(\prod_{i=1}^{m} A_{n+i}\right)\left(\prod_{j=1}^{m'} A_{p'+j}\right)\right]_0 = \left[Q\left(\prod_{i=1}^{m} A_{n+i}\right)\left(\prod_{j=1}^{m'} A_{p'+j}\right)\right]_0 ,$$

$$[P]_0 = [Q]_0 , \qquad a = [P]_0 - [Q]_0 = 0 .$$

■

Remark. By Theorem 8.3.7 and Proposition 8.1.5 c), the functor K_0 is determined by the functor K_1.

COROLLARY 8.3.8 (The six-term sequence) *Let*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) The sequence

$$0 \longrightarrow SF \xrightarrow{S\varphi} SG \xrightarrow{S\psi} SH \longrightarrow 0$$

is exact. Let

$$\delta_2 : K_1(SH) \longrightarrow K_0(SF)$$

be its associated index map (Corollary 7.2.3) *and put* (Proposition 8.1.5, Theorem 7.3.2)

$$\delta_0 := \theta_F^{-1} \circ \delta_2 \circ \beta_H : K_0(H) \longrightarrow K_1(F) .$$

We call δ_0 *and* δ_1 **the six-term index maps**. *If we denote by* $\bar{\delta}_0$ *the corresponding six-term index map associated to the exact sequence in* $\mathfrak{M}_E$ *(with obvious notation)*

$$0 \longrightarrow SF \xrightarrow{\varphi} CF \xrightarrow{\psi} F \longrightarrow 0$$

then $\bar{\delta}_0 = \beta_F$.

b) *The* **six-term sequence**

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) \\ {\scriptstyle\delta_1}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta_0} \\ K_1(H) & \xleftarrow[K_1(\psi)]{} & K_1(G) & \xleftarrow[K_1(\varphi)]{} & K_1(F) \end{array}$$

is exact.

c) *If* F *(resp.* H*) is K-null (e.g. homotopic to* $\{0\}$*) then* $K_i(G) \xrightarrow{K_i(\psi)} K_i(H)$ *(resp.* $K_i(F) \xrightarrow{K_i(\varphi)} K_i(G)$*) is a group isomorphism for every* $i \in \{0,1\}$.

d) *If* G *is K-null (e.g. homotopic to* $\{0\}$*) then*

$$K_0(H) \xrightarrow{\delta_0} K_1(F)\,, \qquad\qquad K_1(H) \xrightarrow{\delta_1} K_0(F)$$

are group isomorphisms.

e) *If* φ *is K-null (e.g. factorizes through null) then the sequences*

$$0 \longrightarrow K_0(G) \xrightarrow{K_0(\psi)} K_0(H) \xrightarrow{\delta_0} K_1(F) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \longrightarrow 0$$

are exact.

f) *If* ψ *is K-null (e.g. factorizes through null) then the sequences*

$$0 \longrightarrow K_0(H) \xrightarrow{\delta_0} K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \longrightarrow 0\,,$$

$$0 \longrightarrow K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \longrightarrow 0$$

are exact.

g) *The six-term index maps of a split exact sequence are equal to* 0.

a) is easy to see.

b) By Theorem 8.3.7, β_H is an isomorphism. By Theorem 7.2.9, the sequences

$$K_1(F) \xrightarrow{K_1(\varphi)} K_1(G) \xrightarrow{K_1(\psi)} K_1(H) \xrightarrow{\delta_1} K_0(F) \xrightarrow{K_0(\varphi)} K_0(G) \xrightarrow{K_0(\psi)} K_0(H)\,,$$

$$K_1(SG) \xrightarrow{K_1(S\psi)} K_1(SH) \xrightarrow{\delta_2} K_0(SF) \xrightarrow{K_0(S\varphi)} K_0(SG)$$

are exact. By Proposition 8.1.5 c) and Proposition 7.3.8, the diagrams

$$\begin{array}{ccc} K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) \\ \beta_G \big\downarrow & & \big\downarrow \beta_H \\ K_1(SG) & \xrightarrow[K_1(S\psi)]{} & K_1(SH) \end{array} \qquad\qquad \begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\varphi)} & K_1(G) \\ \theta_F \big\downarrow & & \big\downarrow \theta_G \\ K_0(SF) & \xrightarrow[K_0(S\varphi)]{} & K_0(SG) \end{array}$$

are commutative. It follows

$$\delta_0 \circ K_0(\psi) = \theta_F^{-1} \circ \delta_2 \circ \beta_H \circ K_0(\psi) = \theta_F^{-1} \circ \delta_2 \circ K_1(S\psi) \circ \beta_G = 0\,,$$

$Im\,K_0(\psi) \subset Ker\,\delta_0$. Let $a \in Ker\,\delta_0$. Then $\delta_2 \beta_H a = \theta_F \delta_0 a = 0$, so there is a $b \in K_1(SG)$ with $K_1(S\psi)\,b = \beta_H a$. It follows

$$a = \beta_H^{-1} K_1(S\psi)\,b = K_0(\psi)\,\beta_G^{-1} b \in Im\,K_0(\psi)\,, \qquad Ker\,\delta_0 \subset Im\,K_0(\psi)\,.$$

c) The assertion follows immediately from b). By Proposition 7.1.8 e), a null-homotopic E-C*-algebra is K-null.

d) The proof is similar to the proof of c).

e) and f) follow from b) and Proposition 7.1.8 f).

g) By Proposition 6.2.9 and Corollary 7.3.9 (with the notation of b)) $K_0(\varphi)$ and $K_1(\varphi)$ are injective and $K_0(\psi)$ and $K_1(\psi)$ are surjective and the assertion follows from b). ■

COROLLARY 8.3.9 *Let us consider the following commutative diagram in $\mathfrak{M}_E$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H & \longrightarrow & 0 \\ & & \gamma \big\downarrow & & \alpha \big\downarrow & & \big\downarrow \beta & & \\ 0 & \longrightarrow & F' & \xrightarrow[\varphi']{} & G' & \xrightarrow[\psi']{} & H' & \longrightarrow & 0\,, \end{array}$$

where the horizontal lines are exact.

a) **(Commutativity of the six-term index maps)** *The diagrams (with obvious notation)*

$$\begin{array}{ccc} K_1(H) & \xrightarrow{\delta_1} & K_0(F) \\ {\scriptstyle K_1(\beta)}\downarrow & & \downarrow{\scriptstyle K_0(\gamma)} \\ K_1(H') & \xrightarrow[\delta_1']{} & K_0(F') \end{array} \qquad \begin{array}{ccc} K_0(H) & \xrightarrow{\delta_0} & K_1(F) \\ {\scriptstyle K_0(\beta)}\downarrow & & \downarrow{\scriptstyle K_1(\gamma)} \\ K_0(H') & \xrightarrow[\delta_0']{} & K_1(F') \end{array}$$

are commutative. If $K_i(F) = K_i(F')$, $K_i(H) = K_i(H')$, and $K_i(\beta)$ and $K_i(\gamma)$ are the identity maps for all $i \in \{0,1\}$ then $\delta_i = \delta_i'$ for all $i \in \{0,1\}$.

b) *The diagram (with obvious notation)*

$$\begin{array}{ccccccccc} K_0(F) & = & K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) & = & K_0(H) \\ = \downarrow & & {\scriptstyle K_0(\gamma)}\downarrow & & {\scriptstyle K_0(\alpha)}\downarrow & & \downarrow{\scriptstyle K_0(\beta)} & & \downarrow = \\ K_0(F) & \xrightarrow{K_0(\gamma)} & K_0(F') & \xrightarrow{K_0(\varphi')} & K_0(G') & \xrightarrow{K_0(\psi')} & K_0(H') & \xleftarrow{K_0(\beta)} & K_0(H) \\ {\scriptstyle \delta_1}\uparrow & & {\scriptstyle \delta_1'}\uparrow & & & & \downarrow{\scriptstyle \delta_0'} & & \downarrow{\scriptstyle \delta_0} \\ K_1(H) & \xrightarrow[K_1(\beta)]{} & K_1(H') & \xleftarrow[K_1(\psi')]{} & K_1(G') & \xleftarrow[K_1(\varphi')]{} & K_1(F') & \xleftarrow[K_1(\gamma)]{} & K_1(F) \\ = \uparrow & & {\scriptstyle K_1(\beta)}\uparrow & & {\scriptstyle K_1(\alpha)}\uparrow & & \uparrow{\scriptstyle K_1(\gamma)} & & \uparrow = \\ K_1(H) & = & K_1(H) & \xleftarrow[K_1(\psi)]{} & K_1(G) & \xleftarrow[K_1(\varphi)]{} & K_1(F) & = & K_1(F) \end{array}$$

is commutative.

a) The commutativity of the first diagram was proved in Proposition 7.2.4. By Proposition 7.3.8, the diagram

$$\begin{array}{ccc} K_1(F) & \xrightarrow{K_1(\gamma)} & K_1(F') \\ {\scriptstyle \theta_F}\downarrow & & \downarrow{\scriptstyle \theta_{F'}} \\ K_0(SF) & \xrightarrow[K_0(S\gamma)]{} & K_0(SF') \end{array}$$

is commutative. By Proposition 7.2.4, the diagram

$$\begin{array}{ccc} K_1(SH) & \xrightarrow{\delta_2} & K_0(SF) \\ {\scriptstyle K_1(S\beta)}\downarrow & & \downarrow{\scriptstyle K_0(S\gamma)} \\ K_1(SH') & \xrightarrow[\delta_2']{} & K_0(SF') \end{array}$$

is commutative, where δ_2 and δ_2' are defined in Corollary 8.3.8 a). By Proposition 8.1.5 c), the diagram

$$\begin{array}{ccc} K_0(H) & \xrightarrow{K_0(\beta)} & K_0(H') \\ {\scriptstyle \beta_H}\downarrow & & \downarrow{\scriptstyle \beta_{H'}} \\ K_1(SH) & \xrightarrow[K_1(S\beta)]{} & K_1(SH') \end{array}$$

is commutative. It follows, by the definition of δ_0 (Corollary 8.3.8 a)),

$$K_1(\gamma)\circ\delta_0 = K_1(\gamma)\circ\theta_F^{-1}\circ\delta_2\circ\beta_H = \theta_{F'}^{-1}\circ K_0(S\gamma)\circ\delta_2\circ\beta_H =$$

$$= \theta_{F'}^{-1}\circ\delta_2'\circ K_1(S\beta)\circ\beta_H = \theta_{F'}^{-1}\circ\delta_2'\circ\beta_{H'}\circ K_0(\beta) = \delta_0'\circ K_0(\beta)\ .$$

b) follows from a) and Corollary 8.3.8 b). ■

Chapter 9

Variation of the Parameters

Throughout this chapter we endow $\{0, 1\}$ with the structure of o group by identifying it with $\mathbf{Z}_2$.

9.1 Changing E

Let E' be a commutative unital C*-algebra, $\phi : E \longrightarrow E'$ a unital C*-homomorphism, and

$$f' : T \times T \longrightarrow Un\,E' , \quad (s,t) \longmapsto \phi f(s,t) .$$

Then $f' \in \mathscr{F}(T,E')$ and we may define E'_n with respect to f' for every $n \in \mathrm{I\!N}$ like in Definition 5.0.2.

Let $n \in \mathrm{I\!N}$ and put

$$C'_n := \sum_{t \in T_n} ((\phi C_{n,t}) \otimes id_K) V_t^{f'} \quad (\in E'_n) .$$

For every $s \in T_{n-1}$,

$$\sum_{t \in T_n} ((f(s^{-1}t,t) C_{n,ts^{-1}}) \otimes id_K) V_t^f = V_s^f C_n =$$

$$= C_n V_s^f = \sum_{t \in T_n} ((f(ts^{-1},s) C_{n,ts^{-1}}) \otimes id_K) V_t^f$$

so by [2] Theorem 2.1.9 a),

$$f(s^{-1}t,t) C_{n,s^{-1}t} = f(ts^{-1},s) C_{n,ts^{-1}}$$

for every $t \in T_n$. It follows

$$f'(s^{-1}t,t) C'_{n,s^{-1}t} = f'(ts^{-1},s) C'_{n,ts^{-1}} , \quad V_s^{f'} C'_n = C'_n V_s^{f'} , \quad C'_n \in (E'_{n-1})^c .$$

Thus $(C'_n)_{n \in \mathrm{I\!N}}$ satisfies the conditions of Axiom 5.0.3 and we may construct a K-theory with respect to T, E', f', and $(C'_n)_{n \in \mathrm{I\!N}}$, which we shall denote by K'.

Let F be an E'-C*-algebra. We denote by $\bar{F}$ or by $\Phi(F)$ the E-C*-algebra obtained by endowing the C*-algebra F with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto (\phi \alpha) x .$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_{E'}$, then $\bar{F} \xrightarrow{\bar{\varphi}} \bar{G}$ is a morphism in $\mathfrak{M}_E$, in a natural way.

Let F be an E'-C*-algebra and $n \in \mathrm{I\!N}$. We put for every

$$X = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in \check{\bar{F}}_n ,$$

$$X' := \sum_{t\in T_n} ((\phi\alpha_t, x_t)\otimes id_K)V_t^{f'} \quad (\in \check{F}_n)$$

and set

$$\phi_{F,n} : \check{\bar{F}}_n \longrightarrow \check{F}_n\,, \quad X \longmapsto X'\ .$$

Then $\phi_{F,n}$ is a unital C*-homomorphism (surjective or injective if ϕ is so ([2] Theorem 2.1.9 a))) such that $\phi_{F,n}(Un_{E_n}\check{\bar{F}}_n) \subset Un_{E'_n}\check{F}_n$ and $\phi_{F,n}\circ\sigma_n^{\bar{F}} = \sigma_n^F\circ\phi_{F,n}$. Thus we get for every $i\in\{0,1\}$ an associated group homomorphism $\Phi_{i,F} : K_i(\bar{F}) \longrightarrow K'_i(F)$.

Let E'' be a unital commutative C*-algebra, $\phi' : E' \longrightarrow E''$ a unital C*-homomorphism, and $\phi'' := \phi'\circ\phi$. Then we may do similar constructions for ϕ' and ϕ'' as we have done for ϕ. If F is an E''-C*-algebra, $\Phi'(F)$ and $\Phi''(F)$ the corresponding E'-C*-algebra and E-C*-algebra, respectively, then $\Phi''(F) = \Phi(\Phi'(F))$. If Φ'_i and Φ''_i are the equivalents of Φ_i with respect to ϕ' and ϕ'', respectively, then $\Phi''_{i,F} = \Phi'_{i,F}\circ\Phi_{i,\Phi'(F)}$ for every $i\in\{0,1\}$. If $E'' = E$ and $\phi'' = id_E$ then $C''_n = C_n$ for every $n\in\mathbb{N}$ and for every E-C*-algebra F, $\Phi''(F) = F$ and $\Phi''_{i,F} = id_{K_i(F)}$ for every $i\in\{0,1\}$. If in addition $\phi''' := \phi\circ\phi' = id_{E'}$ then $C'''_n = C'_n$ for every $n\in\mathbb{N}$ and for every E'-C*-algebra F, $\Phi'(\Phi(F)) = F$ and $\Phi'_{i,\Phi(F)}\circ\Phi_{i,F} = id_{K'_i(F)}$ for every $i\in\{0,1\}$, i.e. the K-theory and the K'-theory "coincide".

Remark. Let $P\in Pr\,E$, $0 < P < 1_E$, and put

$$Pf : T\times T \longrightarrow Un\,PE\,, \quad (s,t)\longmapsto Pf(s,t)\ .$$

Then $Pf\in\mathscr{F}(T,PE)$ and we denote by PK the K-theory with respect to T, PE, Pf, and $(PC_n)_{n\in\mathbb{N}}$. Then for every E-C*-algebra F and $i\in\{0,1\}$

$$K_i(F) \approx ((PK)_i(PF))\times(((1_E - P)K)_i((1_E - P)F))\ .$$

If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_E$ then

$$P\varphi : PF \longrightarrow PG\,, \quad Px\longmapsto P\varphi x$$

is a morphism in $\mathfrak{M}_{PE}$ and

$$K_i(\varphi) = (PK)_i(P\varphi)\times((1_E - P)K)_i((1_E - P)\varphi)$$

for every $i\in\{0,1\}$.

PROPOSITION 9.1.1 *We use the above notation and assume* $i\in\{0,1\}$.

a) If $F \xrightarrow{\varphi} G$ is a morphism in $\mathfrak{M}_{E'}$ then the diagram

$$\begin{array}{ccc} K_i(\bar{F}) & \xrightarrow{K_i(\bar{\varphi})} & K_i(\bar{G}) \\ {\scriptstyle \Phi_{i,F}}\downarrow & & \downarrow{\scriptstyle \Phi_{i,G}} \\ K'_i(F) & \xrightarrow[K'_i(\varphi)]{} & K'_i(G) \end{array}$$

is commutative.

b) For every E'-C-algebra F the diagram*

$$\begin{array}{ccc} K_0(\bar{F}) & \xrightarrow{\beta_{\bar{F}}} & K_1(\overline{SF}) \\ {\scriptstyle \Phi_{0,F}}\downarrow & & \downarrow{\scriptstyle \Phi_{1,SF}} \\ K'_0(F) & \xrightarrow[\beta'_F]{} & K'_1(SF)\,, \end{array}$$

is commutative, where β'_F denotes the Bott map in the K'-theory.

c) If

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_{E'}$ then the diagram

$$\begin{array}{ccc} K_1(\bar{H}) & \xrightarrow{\delta_1} & K_0(\bar{F}) \\ {\scriptstyle \Phi_{1,H}}\downarrow & & \downarrow{\scriptstyle \Phi_{0,F}} \\ K'_1(H) & \xrightarrow[\delta'_1]{} & K'_0(F) \end{array}$$

is commutative, where δ'_1 denotes the index maps associated to the above exact sequences in the K'-theory.

a) For every $n \in \mathbb{N}$ and

$$X = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in \check{\bar{F}}_n\,,$$

$$\check{\varphi}_n \phi_{F,n} X = \sum_{t \in T_n} (((\phi\,\alpha_t), \varphi x_t) \otimes id_K) V_t^{f'} = \phi_{G,n} \check{\bar{\varphi}}_n X\,.$$

b) For every $n \in \mathbb{N}$ and $P \in Pr\,\check{\bar{F}}_n$,

$$\phi_{SF,n} \widetilde{P} = (\widetilde{P})' = \widetilde{P'} = \widetilde{\phi_{F,n} P}\,.$$

c) Let $n \in \mathbb{N}$ and $U \in Un\, \check{\check{H}}_{n-1}$. By Proposition 7.2.1 a), there are $V \in Un\, \check{\check{G}}_n$ and $P \in Pr\, \check{\check{F}}_n$ such that

$$\check{\check{\psi}}_n V = A_n U + B_n U^* , \qquad \check{\check{\varphi}}_n P = V A_n V^* .$$

Then

$$\check{\psi}_n \phi_{G,n} V = \phi_{H,n} \check{\check{\psi}}_n V = A'_n(\phi_{H,n-1} U) + B'_n(\phi_{H,n-1} U)^* ,$$

$$\check{\varphi}_n \phi_{F,n} P = \phi_{G,n} \check{\check{\varphi}}_n P = (\phi_{G,n} V) A'_n (\phi_{G,n} V)^*$$

so by Corollary 7.2.3,

$$\delta'_1 \Phi_{1,H}[U]_1 = \delta'_1[\phi_{H,n-1}U]_1 = [\phi_{F,n}P]_0 = \Phi_{0,F}[P]_0 = \Phi_{0,F}\delta_1[U]_1$$

$$\delta'_1 \circ \Phi_{1,H} = \Phi_{0,F} \circ \delta_1 . \qquad \blacksquare$$

LEMMA 9.1.2 *Let F, G be C*-algebras, $\varphi : F \longrightarrow G$ a surjective C*-homomorphism, and*

$$\psi : \mathscr{C}([0,1],F) \longrightarrow \mathscr{C}([0,1],G) , \quad x \longmapsto \varphi \circ x .$$

a) ψ is surjective.

b) Assume F unital and let $v \in Un\, \mathscr{C}([0,1],G)$ such that there is an $x \in Un\, F$ with $\varphi x = v(0)$. Then there is a $u \in Un\, \mathscr{C}([0,1],F)$ with $\psi u = v$ and $u(0) = x$.

a) Let y be an element of $\mathscr{C}([0,1],G)$ which is piecewise linear, i.e. there is a family

$$0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$$

such that for every $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$,

$$y((1-t)s_i + ts_{i+1}) = (1-t)y(s_i) + ty(s_{i+1}) .$$

Since φ is surjective, there is a family $(x_i)_{i\in\mathbb{N}_n}$ in F with $\varphi x_i = y(s_i)$ for every $i \in \mathbb{N}_n$. Define $x : [0,1] \longrightarrow F$ by putting

$$x((1-t)s_i + ts_{i+1}) := (1-t)x_i + tx_{i+1}$$

for every $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$. For $i \in \mathbb{N}_{n-1}$ and $t \in [0,1]$,

$$(\psi x)((1-t)s_i + ts_{i+1}) = \varphi((1-t)x_i + tx_{i+1}) =$$

$$= (1-t)y(s_i) + ty(s_{i+1}) = y((1-t)s_i + ts_{i+1}) \,,$$

so $\psi x = y$, $y \in Im\,\psi$. Since the set of elements of $\mathscr{C}([0,1],G)$, which are piecewise linear, is dense in $\mathscr{C}([0,1],G)$ and $Im\,\psi$ is closed (as C*-homomorphism), ψ is surjective.

b) Let

$$w : [0,1] \longrightarrow Un\,G\,, \quad s \longmapsto v(0)^* v(s)\,.$$

Then $w \in Un\,\mathscr{C}([0,1],G)$ and $w(0) = 1_G$. Put

$$w_t : [0,1] \longrightarrow Un\,G\,, \quad s \longmapsto w(st)$$

for every $t \in [0,1]$. Then

$$[0,1] \longrightarrow Un\,\mathscr{C}([0,1],G)\,, \quad t \longmapsto w_t$$

is a continuous path with $w_1 = w$ and $w_0 = 1_{\mathscr{C}([0,1],G)}$. Thus

$$w \in Un_0\,\mathscr{C}([0,1],G)\ .$$

By a), ψ is surjective, so by [4] Lemma 2.1.7 (i), there is a $u' \in Un\,\mathscr{C}([0,1],F)$ with $\psi u' = w$. Put

$$u : [0,1] \longrightarrow Un\,F\,, \quad s \longmapsto xu'(0)^* u'(s)\,.$$

Then $u \in Un\,\mathscr{C}([0,1],F)$, $u(0) = x$, and

$$(\psi u)(s) = \varphi(u(s)) = \varphi(xu'(0)^* u'(s)) = \varphi(x)((\psi u')(0))^*((\psi u')(s)) =$$

$$= v(0)w(0)^* w(s) = v(0)1_G v(0)^* v(s) = v(s)$$

for every $s \in [0,1]$, i.e. $\psi u = v$. ■

THEOREM 9.1.3 *$\Phi_{i,F}$ is a group isomorphism for every $i \in \{0,1\}$ and for every E'-C*-algebra F.*

By Proposition 9.1.1 b), $\Phi_{0,F} = (\beta'_F)^{-1} \circ \Phi_{1,SF} \circ \beta_{\bar{F}}$, so it suffices to prove the assertion for $\Phi_{1,F}$ only. Let $n \in \mathbb{N}$ and $U \in Un\,\check{F}_n$. Put $V := U(\sigma_n^F U)^* \sim_1 U$. Since $\sigma_n^F V = 1_{E'}$, V has the form

$$V = \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^{f'}$$

with $\alpha_t = \delta_{1,t} 1_{E'}$ and $x_t \in F$ for every $t \in T_n$. If we put

$$W := \sum_{t \in T_n} ((\delta_{1,t} 1_E, x_t) \otimes id_K) V_t^f$$

then $\phi_{F,n} W = V$ and we get $\Phi_{1,F}[W]_1 = [V]_1 = [U]_1$, so $\Phi_{1,F}$ is surjective. Thus we have to prove the injectivity of $\Phi_{1,F}$ only.

Let $a \in Ker\, \Phi_{1,F}$. We have to prove $a = 0$. There are $n \in \mathbb{N}$ and

$$U := \sum_{t \in T_n} ((\alpha_t, x_t) \otimes id_K) V_t^f \in Un\, \check{\check{F}}_n$$

with $a = [U]_1$, where $(\alpha_t, x_t) \in \check{F}$ for every $t \in T_n$. Since $[U']_1 = \Phi_{1,F}[U]_1 = 0$, by Proposition 7.1.3, there is an $m \in \mathbb{N}$ such that

$$U'_0 := \left(\prod_{i=1}^{m} A'_{n+i} \right) U' + \left(1_{E'} - \prod_{i=1}^{m} A'_{n+i} \right)$$

is homotopic in $Un\, \check{F}_{n+m}$ to a $U'_1 \in Un\, E'_{n+m}$ $(\subset Un\, \check{F}_{n+m})$. Thus there is a continuous path

$$U' : [0,1] \longrightarrow Un\, \check{F}_{n+m}, \quad s \longmapsto U'_s\,.$$

Case 1 ϕ is injective

Put

$$W'_s := U'_s \sigma^F_{n+m} (U'^*_s U'_0)\ (\in Un\, \check{F}_{n+m})$$

for every $s \in [0,1]$. Then

$$\sigma^F_{n+m} W'_s = \sigma^F_{n+m} U'_0 = \phi_{F,n+m} \left(\left(\prod_{i=1}^{m} A_{n+i} \right) (\sigma^{\bar{F}}_n U) + \left(1_E - \prod_{i=1}^{m} A_{n+i} \right) \right)$$

for every $s \in [0,1]$. If we put

$$W'_s =: \sum_{t \in T_{n+m}} ((\beta_{s,t}, y_{s,t}) \otimes id_K) V_t^{f'}\,,$$

where $(\beta_{s,t}, y_{s,t}) \in \check{F}$ for all $s \in [0,1]$ and $t \in T_n$, then

$$\sum_{t \in T_{n+m}} ((\beta_{s,t}, 0) \otimes id_K) V_t^{f'} = \sigma^F_{n+m} W'_s =$$

$$= \phi_{F,n+m}\left(\left(\prod_{i=1}^{m} A_{n+i}\right)\sum_{t\in T_n}((\alpha_t,0)\otimes id_K)V_t^f + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right)$$

and so by [2] Theorem 2.1.9 a), there is a (unique) family $(\gamma_t)_{t\in T_{n+m}}$ in E with $\beta_{s,t} = \phi\gamma_t$ for every $s \in [0,1]$ and $t \in T_{n+m}$. Since ϕ is injective, ϕ_{n+m} is also injective and $\phi_{n+m}(\check{\check{F}}_{n+m})$ may be identified with a unital C*-subalgebra of $\check{F}_{n+m}$. Thus

$$W : [0,1] \longrightarrow Un\,\check{\check{F}}_{n+m}\,, \quad s \longmapsto \sum_{t\in T_{n+m}}((\gamma_t, y_{s,t})\otimes id_K)V_t^f$$

is a continuous path in $Un\,\check{\check{F}}_{n+m}$ with $\phi_{F,n+m}W_s = W'_s$ for every $s \in [0,1]$. It follows

$$\phi_{F,n+m}W_0 = W'_0 = U'_0 = \phi_{F.,n+m}\left(\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\right),$$

$$\phi_{F,n+m}W_1 = W'_1 = U'_1\sigma^F_{n+m}(U'^*_1 U'_0) = \sigma^F_{n+m}U'_0 \in \phi_{F,n+m}(Un\,E'_{n+m})\,.$$

Since ϕ is injective, $\phi_{F,n+m}$ is also injective and we get

$$\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) = W_0\,,$$

$$\left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right) \in Un_{E_{n+m}}\check{\check{F}}_{n+m}\,, \qquad g = [U]_1 = 0\,.$$

Case 2 ϕ is surjective

We put

$$\bar{U}_0 := \left(\prod_{i=1}^{m} A_{n+i}\right)U + \left(1_E - \prod_{i=1}^{m} A_{n+i}\right)\ (\in Un\,\check{\check{F}}_{n+m})\,.$$

Since ϕ is surjective, $\phi_{F,n+m}$ is also surjective ([2] Theorem 2.1.9 a)). Since

$$\phi_{F,n+m}\bar{U}_0 = U'_0$$

it follows from Lemma 9.1.2 b), that there is a continuous path

$$[0,1] \longrightarrow Un\,\check{\check{F}}_{n+m}, \quad s \longmapsto U_s$$

with $\phi_{F,n+m}U_s = U'_s$ for every $s \in [0,1]$ and $U_0 = \bar{U}_0$. Since $\phi_{F,n+m}U_1 = U'_1 \in Un\,E'_{n+m}$, we have $\bar{U}_0 \in Un_{E_{n+m}}\check{\check{F}}_{n+m}$ and $g = [U]_1 = [\bar{U}_0]_1 = 0$.

Case 3 ϕ is arbitrary

There are a unital commutative C*-algebra E'' and a unital C*-homomor-phisms $\phi' : E \longrightarrow E''$ and $\phi'' : E'' \longrightarrow E'$ such that ϕ' is surjective, ϕ'' is injective, and $\phi = \phi'' \circ \phi'$ and the assertion follows from the first two cases and the considerations from the begin of the section. ■

COROLLARY 9.1.4 *Let E', E'' be unital commutative C*-algebras such that $E = E' \times E''$ and*

$$\phi' : E \longrightarrow E', \quad (x', x'') \longmapsto x',$$

$$\phi'' : E \longrightarrow E'', \quad (x', x'') \longmapsto x''.$$

If F' is an E'-C-algebra and F'' is an E''-C*-algebra then the map (with obvious notation)*

$$K_i(\Phi'(F') \times \Phi''(F'')) \longrightarrow K_i'(F') \times K_i''(F''), \quad a \longmapsto (\Phi'_{i,F'} \times \Phi''_{i,F''})(\varphi_i a)$$

is a group isomorphism for every $i \in \{0,1\}$, where

$$\varphi_i : K_i(\Phi'(F') \times \Phi''(F'')) \longrightarrow K_i(\Phi'(F')) \times K_i(\Phi''(F''))$$

is the canonical group isomorphism (Product Theorem (Corollary 6.2.10 b), Proposition 7.3.3 b)). ■

COROLLARY 9.1.5 *If $f(s,t) \in \mathbb{C}$ for all $s,t \in T$ and $C_n \in \mathbb{C}_n$ for all $n \in \mathbb{N}$and if $K^{\mathbb{C}}$ denotes the K-theory with respect to T, $\mathbb{C}$, f, and $(C_n)_{n\in\mathbb{N}}$ then $K_i(E) = K_i^{\mathbb{C}}(\mathscr{C}(\Omega, \mathbb{C}))$ for all $i \in \{0,1\}$, where Ω denotes the spectrum of E.* ■

PROPOSITION 9.1.6 *If F is an E'-C*-algebra then the map*

$$\varphi : E \times \Phi(F) \longrightarrow \overset{\vee}{\overbrace{\Phi(F)}}, \quad (\alpha, x) \longmapsto (\alpha, x - \phi\alpha)$$

is an E-C-isomorphism.*

For $(\alpha, x), (\beta, y) \in E \times \Phi(F)$ and $\gamma \in E$,

$$\varphi(\gamma(\alpha, x)) = \varphi(\gamma\alpha, (\phi\gamma)x) = (\gamma\alpha, (\phi\gamma)x - \phi(\gamma\alpha)) =$$

$$= (\gamma,0)(\alpha,x-\phi\alpha) = (\gamma,0)\varphi(\alpha,x)\,,$$

$$\varphi(\alpha,x)^* = \varphi(\alpha^*,x^*) = (\alpha^*,x^*-\phi\alpha^*) = (\varphi(\alpha,x))^*\,,$$

$$\varphi(\alpha,x)\varphi(\beta,y) = (\alpha,x-\phi\alpha)(\beta,y-\phi\beta) =$$

$$= (\alpha\beta,(\phi\alpha)y-\phi(\alpha\beta)+(\phi\beta)x-\phi(\alpha\beta)+xy-(\phi\beta)x-(\phi\alpha)y+\phi(\alpha\beta)) =$$

$$= (\alpha\beta,xy-\phi(\alpha\beta)) = \varphi(\alpha\beta,xy) = \varphi((\alpha,x)(\beta,y))\,,$$

so φ is an E-C*-homomorphism. The other assertions are easy to see. ■

9.2 Changing f

In all Propositions and Corollaries of this section we use the notation and assumptions of Example 5.0.4 and F denotes a C*-algebra.

LEMMA 9.2.1 *For every $n \in \mathbb{N}$ there is an $\varepsilon_n > 0$ such that for every $m \in \mathbb{N}$, $m \le n$, and $\alpha \in Un\,\mathbb{C}$, $|\alpha - 1| < \varepsilon_n$, there is a unique $\beta_\alpha \in Un\,\mathbb{C}$, $|\beta_\alpha - 1| < \frac{1}{n}$, with $\beta_\alpha^m = \alpha$; moreover the map $\alpha \mapsto \beta_\alpha$ is continuous.*

If β,γ are distinct elements of $Un\,\mathbb{C}$ and $\beta^m = \gamma^m$ then

$$|\beta-\gamma| \ge |1-e^{\frac{2\pi i}{m}}| > \frac{1}{m} \ge \frac{1}{n}$$

and the assertion follows from the continuity of the corresponding branch of the map $\alpha \mapsto \sqrt[m]{\alpha}$. ■

DEFINITION 9.2.2 *For every finite group S we endow $\mathscr{F}(S,\mathbb{C})$ with the metric*

$$d_S(g,h) := \sup\{\, |g(s,t)-h(s,t)| \mid s,t \in S\}$$

for all $g,h \in \mathscr{F}(S,\mathbb{C})$.

Remark. $\mathscr{F}(S,\mathbb{C})$ endowed with the above metric is compact.

DEFINITION 9.2.3 *We put*

$$\Lambda(T,E) := \{\, \lambda : T \longrightarrow Un\, E \mid \lambda(1) = 1_E \,\}$$

and

$$\delta\lambda : T \times T \longrightarrow Un\, E\,, \quad (s,t) \longmapsto \lambda(s)\lambda(t)\lambda(st)^*$$

for every $\lambda \in \Lambda(T,E)$.

LEMMA 9.2.4 *Let S be a finite group and Ω a compact space.*

a) $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbb{C}) \,\}$ is an open set of $\mathscr{F}(S,\mathbb{C})$.

b) For every $\varepsilon' > 0$ there is an $\varepsilon > 0$ such that for all $g,h \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbb{C}))$, if

$$\|g(s,t) - h(s,t)\| < \varepsilon$$

for all $s,t \in S$ then there is a $\lambda \in \Lambda(S,\mathbb{C})$ such that $h = g\delta\lambda$ and $|\lambda(s) - 1| < \varepsilon'$ for all $s \in S$.

c) Let $g \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbb{C}))$ and $\phi : [0,1] \times \Omega \longrightarrow \Omega$ a continuous map. We put for every $u \in [0,1]$,

$$\phi_u := \phi(u,\cdot) : \Omega \longrightarrow \Omega\,,$$

$$g_u : S \times S \longrightarrow Un\,\mathbb{C}\,, \quad (s,t) \longmapsto g(s,t) \circ \phi_u\,.$$

Then $g_u \in \mathscr{F}(S,\mathscr{C}(\Omega,\mathbb{C}))$ for every $u \in [0,1]$ and there is a $\lambda \in \Lambda(S,\mathbb{C})$ with $g_1 = g_0\delta\lambda$.

a) By [3] Theorem 2.3.2 (iii),

$$\{\, \mathscr{S}(g) \mid g \in \mathscr{F}(S,\mathbb{C}) \,\} / \approx_{\mathscr{S}}$$

is finite. $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbb{C}) \,\}$ is obviously a closed subgroup of $\mathscr{F}(S,\mathbb{C})$. By the above and [2] Proposition 2.2.2 c), $\mathscr{F}(S,\mathbb{C})$ is the union of a finite family of closed pairwise disjoint sets homeomorphic to $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbb{C}) \,\}$, so $\{\, \delta\lambda \mid \lambda \in \Lambda(S,\mathbb{C}) \,\}$ is open.

b) By a), there is an $\varepsilon > 0$ such that for all $g',h' \in \mathscr{F}(S,\mathbb{C})$ with $d_S(g',h') < \varepsilon$ there is a $\lambda \in \Lambda(S,\mathbb{C})$ with $h' = g'\delta\lambda$. We may assume that

$$(1+\varepsilon)^{Card\, S} - 1 < \varepsilon_{Card\, S}\,,$$

where $\varepsilon_{Card\,S}$ was defined in Lemma 9.2.1.

We put for every $\omega \in \Omega$

$$g_\omega : S \times S \longrightarrow Un\,\mathbf{C}, \quad (s,t) \longmapsto (g(s,t))(\omega),$$

$$h_\omega : S \times S \longrightarrow Un\,\mathbf{C}, \quad (s,t) \longmapsto (h(s,t))(\omega).$$

Let $\omega \in \Omega$. By the above, there is a $\lambda_\omega \in \Lambda(S,\mathbf{C})$ with $g_\omega = h_\omega \delta \lambda_\omega$. Let $s \in S$ and let $n \in \mathbb{N}$ be the least natural number with $s^n = 1_S$. By [2] Proposition 3.4.1 c),

$$\lambda_\omega(s)^n = \prod_{j=1}^{n-1} (g_\omega(s^j,s)^* h_\omega(s^j,s)).$$

For every $j \in \mathbb{N}_{n-1}$,

$$\left\| 1_E - g(s^j,s)^* h(s^j,s) \right\| = \left\| g(s^j,s) - h(s^j,s) \right\| < \varepsilon,$$

$$\left\| \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)) \right\| = \left\| \prod_{j=1}^{n-1} (1_E - (1_E - g(s^j,s)^* h(s^j,s))) \right\| < (1+\varepsilon)^n,$$

$$\left\| 1_E - \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)) \right\| < (1+\varepsilon)^{n-1} - 1 < \varepsilon_{Card\,S}.$$

By Lemma 9.2.1, there is a unique $\gamma \in Un\,\mathbf{C}$ with

$$\gamma^n = \prod_{j=1}^{n-1} (g(s^j,s)^* h(s^j,s)), \qquad |\gamma - 1| < \frac{1}{Card\,S}.$$

For $\omega \in \Omega$, since $|1 - \lambda_\omega(s)| < \varepsilon_{Card\,S}$, we get $\lambda_\omega(s) = \gamma(s)$. So if we put

$$\lambda(s) : \Omega \longrightarrow \mathbf{C}, \quad \omega \longmapsto \gamma(s)$$

we have $\lambda \in \Lambda(S,\mathbf{C})$ and $g = h\delta\lambda$. By Lemma 9.2.1, we may choose ε in such a way that the inequality $|\lambda(s) - 1| < \varepsilon'$ holds for all $s \in S$.

c) By b), there is a family $(\lambda_i)_{i \in \mathbb{N}_n}$ in $\Lambda(S,\mathbf{C})$ and

$$0 = u_0 < u_1 < \cdots < u_{n-1} < u_n = 1$$

such that $g_{u_i} = g_{u_{i-1}} \delta \lambda_i$ for every $i \in \mathbb{N}_n$. By induction $g_0 \delta \left(\prod_{i=1}^{j} \lambda_i \right) = g_{u_j}$ for every $j \in \mathbb{N}_n$. Thus if we put $\lambda := \prod_{i=1}^{n} \lambda_i$ then $g_0 \delta \lambda = g_1$ ■

Remark. Let $\lambda \in \Lambda(T,E)$ and $f' = f\delta\lambda$ $(\in \mathscr{F}(T,E))$. For every full E-C*-algebra F and $n \in \mathrm{I\!N}$ we denote by F'_n the equivalent of F_n constructed with respect to f' instead of f (Definition 5.0.2). By [2] Proposition 2.2.2 $a_1 \Rightarrow a_2$, there is for every $n \in \mathrm{I\!N}$ a unique E-C*-isomorphism $\varphi_n^F : F_n \longrightarrow F'_n$ such that for all $m,n \in \mathrm{I\!N}$, $m < n$, the diagram

$$\begin{tikzcd} F_m \arrow[r, "\varphi_m^F"] \arrow[d] & F'_m \arrow[d] \\ F_n \arrow[r, "\varphi_n^F"'] & F'_n \end{tikzcd}$$

is commutative, where the vertical arrows are the canonical inclusions. We put $C'_n := \varphi_n^E C_n$ for evrey $n \in \mathrm{I\!N}$. $(C'_n)_{n\in\mathrm{I\!N}}$ satisfies the conditions of Axiom 5.0.3 with respect to f', so we can construct a K-theory with respect to T, E, f', and $(C'_n)_{n\in\mathrm{I\!N}}$, which we shall denote by $K^{f'}$. If $m,n \in \mathrm{I\!N}$, $m < n$, then the diagrams

$$\begin{tikzcd} F_m \arrow[r, "\rho_{n,m}^F"] \arrow[d, "\varphi_m^F"'] & F_n \arrow[d, "\varphi_n^F"] \\ F'_m \arrow[r, "\rho_{n,m}^{F'}"'] & F'_n \end{tikzcd} \qquad \begin{tikzcd} Un\,F_m \arrow[r, "\tau_{n,m}^F"] \arrow[d, "\varphi_m^F"'] & Un\,F_n \arrow[d, "\varphi_n^F"] \\ Un\,F'_m \arrow[r, "\tau_{n,m}^{F'}"'] & Un\,F'_n \end{tikzcd}$$

are commutative and so we get the isomorphisms

$$Pr\,F_{\rightarrow} \longrightarrow Pr\,F'_{\rightarrow}\,, \qquad\qquad un\,F_{\leftarrow} \longrightarrow un\,F'_{\leftarrow}\,.$$

By these considerations it can be followed that K and $K^{f'}$ coincide.

DEFINITION 9.2.5 *Let Ω be the spectrum of E, Γ a closed set of Ω, and F a C*-algebra. We denote by $\mathscr{C}(E;\Gamma,F)$ the E-C*-algebra obtained by endowing the C*-algebra $\mathscr{C}(\Gamma,F)$ with the structure of an E-C*-algebra by putting*

$$\alpha x : \Gamma \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x(\omega)$$

for all $(\alpha,x) \in E \times \mathscr{C}(\Gamma,F)$. If Ω' is an open set of Ω then the ideal and E-C-subalgebra*

$$\left\{\, x \in \mathscr{C}(E;\Omega,F) \mid x|(\Omega \setminus \Omega') = 0 \,\right\}$$

of $\mathscr{C}(E;\Omega,F)$ will be denoted $\mathscr{C}_0(E;\Omega',F)$.

By Tietze's theorem

$$0 \longrightarrow \mathscr{C}_0\left(E;\Omega',F\right) \xrightarrow{\varphi} \mathscr{C}\left(E;\Omega,F\right) \xrightarrow{\psi} \mathscr{C}\left(E;\Omega\setminus\Omega',F\right) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$, where φ denotes the inclusion map and

$$\psi : \mathscr{C}\left(E;\Omega,F\right) \longrightarrow \mathscr{C}\left(E;\Omega\setminus\Omega',F\right), \quad x \longmapsto x|(\Omega\setminus\Omega') .$$

PROPOSITION 9.2.6 *We denote by Ω the spectrum of E, by Γ a closed set of Ω, and by $\vartheta : [0,1]\times\Omega \longrightarrow \Omega$ a continuous map such that*

$$\omega\in\Omega \Longrightarrow \vartheta(0,\omega)=\omega,\ \vartheta(1,\omega)\in\Gamma$$

and $\vartheta(s,\omega)=\omega$ for all $s\in[0,1]$ and $\omega\in\Gamma$. We put $E':=\mathscr{C}\left(\Gamma,\mathbb{C}\right)$, $E'':=E$, $\vartheta_s:=\vartheta(s,\cdot)$ for every $s\in[0,1]$, and

$$\phi : E \longrightarrow E', \quad x\longmapsto x|\Gamma, \qquad\qquad \phi' : E' \longrightarrow E''=E, \quad x'\longmapsto x'\circ\vartheta_1,$$

$$f' : T\times T \longrightarrow Un\,E', \quad (s,t)\longmapsto \phi f(s,t)=f(s,t)|\Gamma,$$

$$f'' : T\times T \longrightarrow Un\,E'', \quad (s,t)\longmapsto \phi' f'(s,t)=f(s,t)\circ\vartheta_1 .$$

a) *There is a $\lambda\in\Lambda(T,E)$ such that $f''=f\delta\lambda$ and the K-theories associated to f and f'' coincide (as formulated in the above* Remark*). If Γ is a one-point set (i.e. Ω is contractible) then $f''(s,t)\in Un\,\mathbb{C}$ $(\subset Un\,E)$ for all $s,t\in T$.*

b) *If we put*

$$\psi : \mathscr{C}\left(E;\Omega,F\right) \longrightarrow \mathscr{C}\left(E;\Gamma,F\right), \quad x\longmapsto x|\Gamma$$

then $K_i(\mathscr{C}_0\left(E;\Omega\setminus\Gamma,F\right))=\{0\}$ and

$$K_i(\psi) : K_i(\mathscr{C}\left(E;\Omega,F\right)) \longrightarrow K_i(\mathscr{C}\left(E;\Gamma,F\right))$$

is a group isomorphism for every $i\in\{0,1\}$.

c) *If Γ' is a compact subspace of $\Omega\setminus\Gamma$ then*

$$K_i(\mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\Gamma'),F\right)) \approx K_{i+1}(\mathscr{C}\left(E;\Gamma',F\right))$$

for all $i\in\{0,1\}$.

d) Let $\bar{\Gamma}$ be a closed set of Ω, $\bar{\varphi} : \mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\bar{\Gamma}),F\right) \longrightarrow \mathscr{C}\left(E;\Omega,F\right)$ the inclusion map,

$$\bar{\psi} : \mathscr{C}_0\left(E;\Omega,F\right) \longrightarrow \mathscr{C}\left(E;\Gamma\cup\bar{\Gamma},F\right), \quad x \longmapsto x|(\Gamma\cup\bar{\Gamma}),$$

and δ_0,δ_1 the corresponding maps from the six-term sequence associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\bar{\Gamma}),F\right) \xrightarrow{\bar{\varphi}} \mathscr{C}\left(E;\Omega,F\right) \xrightarrow{\bar{\psi}} \mathscr{C}\left(E;\Gamma\cup\bar{\Gamma},F\right) \longrightarrow 0$$

then the sequence

$$0 \longrightarrow K_i(\mathscr{C}\left(E;\Omega,F\right)) \xrightarrow{K_i(\psi)} K_i(\mathscr{C}\left(E;\Gamma\cup\bar{\Gamma},F\right)) \xrightarrow{\delta_i}$$

$$\xrightarrow{\delta_i} K_{i+1}(\mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\bar{\Gamma}),F\right)) \longrightarrow 0$$

is exact for every $i\in\{0,1\}$.

a) By Lemma 9.2.4 c), for every $m\in\mathbb{N}$ there is a $\lambda_m\in\Lambda(S_m,E)$ with $f''|(S_m\times S_m) = g_m\delta\lambda_m$. We put

$$\lambda : T \longrightarrow Un\,E, \quad t\longmapsto \lambda_m(t) \quad \text{if} \quad t\in S_m\,.$$

Then

$$f''(s,t) = \prod_{m\in\mathbb{N}}(g_m\delta\lambda)(s_m,t_m) = (f\delta\lambda)(s,t)$$

for all $s,t\in T$, i.e. $f'' = f\delta\lambda$.

b) Let $n\in\mathbb{N}$ and $X\in\left(\overbrace{\mathscr{C}_0\left(E'';\Omega\setminus\Gamma,F\right)}^{\vee}\right)_n$. Then X has the form

$$X = \sum_{t\in T_n}((\alpha_t,x_t)\otimes id_K)V_t^{f''},$$

where $\alpha_t\in E''$ and $x_t\in\mathscr{C}_0\left(E'';\Omega\setminus\Gamma,F\right)$ for all $t\in T_n$. We put

$$X_s := \sum_{t\in T_n}((\alpha_t\circ\vartheta_s, x_t\circ\vartheta_s)\otimes id_K)V_t^{f''}$$

for every $s\in[0,1]$. Then

$$[0,1] \longrightarrow \left(\overbrace{\mathscr{C}_0\left(E'';\Omega\setminus\Gamma,F\right)}^{\vee}\right)_n, \quad s\longmapsto X_s$$

is a continuous map, $X_0 = X$,

$$X_1 = \sum_{t\in T}((\alpha_t \circ \vartheta_1, 0) \otimes id_K)V_t^{f''} ,$$

and

$$\left(\overbrace{\mathscr{C}_0\left(E'' : \Omega\setminus\Gamma, F\right)}^{\vee}\right)_n \longrightarrow \left(\overbrace{\mathscr{C}_0\left(E''; \Omega\setminus\Gamma, F\right)}^{\vee}\right)_n , \quad X \longmapsto X_s$$

is an E''-C*-homomorphism for every $s \in [0,1]$. Thus $K_i^{f''}(\mathscr{C}_0(E''; \Omega\setminus\Gamma, F)) = \{0\}$. By a), $K_i(\mathscr{C}_0(E; \Omega\setminus\Gamma, F)) = \{0\}$.

If $\varphi : \mathscr{C}_0(E;\Omega\setminus\Gamma, f) \longrightarrow \mathscr{C}(E;\Omega, F)$ denotes the inclusion map then

$$0 \longrightarrow \mathscr{C}_0(E; \Omega\setminus\Gamma, F) \xrightarrow{\varphi} \mathscr{C}(E; \Omega, F) \xrightarrow{\psi} \mathscr{C}(E; \Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and the assertion follows from the six-term sequence (Corollary 8.3.8 c)).

c) If we put

$$F_1 := \mathscr{C}_0\left(E; \Omega\setminus(\Gamma\cup\Gamma'), F\right), \quad F_2 := \mathscr{C}_0(E; \Omega\setminus\Gamma, F), \quad F_3 := \mathscr{C}\left(E; \Gamma', F\right),$$

$$\varphi : F_1 \longrightarrow F_2, \quad x \longmapsto x,$$

$$\psi : F_2 \longrightarrow F_3, \quad x \longmapsto x|\Gamma'$$

then

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and the assertion follows from b) and from the six-term sequence (Corollary 8.3.8 d)).

d) $\bar{\varphi}$ factorizes through $\mathscr{C}_0(E; \Omega\setminus\Gamma, f)$ so by b), $K_i(\bar{\varphi}) = 0$ and the assertion follows from the six-term sequence Corollary 8.3.8 b). ■

COROLLARY 9.2.7 *We use the notation of* Proposition 9.2.6. *Let $\bar{\Omega}$ be a compact space and $\bar{\vartheta} : \Omega \longrightarrow \bar{\Omega}$ a continuous map such that the induced maps $\Omega\setminus(\Gamma\cup\Gamma') \to \bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma')$, $\Gamma \to \bar{\vartheta}(\Gamma)$, and $\Gamma' \to \bar{\vartheta}(\Gamma')$ are homeomorphisms. If we put $\bar{E} := \mathscr{C}\left(\bar{\Omega}, \mathbb{C}\right)$ and*

$$\bar{\phi} : \bar{E} \longrightarrow E, \quad x \longmapsto x\circ\bar{\vartheta}$$

and take an $\bar{f} \in \mathscr{F}(T,\bar{E})$ *such that* $f(s,t) = \bar{\phi}\bar{f}(s,t)$ *for all* $s,t \in T$ *and a corresponding* $(\bar{C}_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} \bar{E}_n$ *then with the notation from the beginning of* section 9.1 (with E and $\bar{E}$ interchanged)

$$\bar{K}_i\left(\mathscr{C}_0\left(\bar{E};\bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'),F\right)\right) \approx \bar{K}_{i+1}\left(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma'),F\right)\right),$$

for all $i \in \{0,1\}$*, where* $\bar{K}$ *denotes the K-theory associated to* T*,* $\bar{E}$*,* $\bar{f}$*, and* $(\bar{C}_n)_{n\in\mathbb{N}}$*. If in addition* Γ' *has the same property as* Γ *then*

$$\bar{K}_i\left(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma),F\right)\right) \approx \bar{K}_i\left(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma'),F\right)\right).$$

By our hypotheses,

$$\bar{\Phi}\left(\mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\Gamma'),F\right)\right) \approx \mathscr{C}_0\left(\bar{E};\bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'),F\right),$$

$$\bar{\Phi}(\mathscr{C}(E;\Gamma,F)) \approx \mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma),F\right), \qquad \bar{\Phi}\left(\mathscr{C}\left(E;\Gamma',F\right)\right) \approx \mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma'),F\right),$$

so by Proposition 9.2.6 b) and Theorem 9.1.3,

$$\bar{K}_i\left(\mathscr{C}_0\left(\bar{E};\bar{\Omega}\setminus\bar{\vartheta}(\Gamma\cup\Gamma'),F\right)\right) \approx K_i\left(\mathscr{C}_0\left(E;\Omega\setminus(\Gamma\cup\Gamma'),F\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}\left(E;\Gamma',F\right)\right) \approx \bar{K}_{i+1}(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma'),F\right)).$$

If the supplementary hypothesis is fulfilled then by Proposition 9.2.6 c) and Theorem 9.1.3,

$$\bar{K}_i\left(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma),F\right)\right) \approx K_i(\mathscr{C}(E;\Gamma,F)) \approx$$

$$\approx K_i\left(\mathscr{C}\left(E;\Gamma'),F\right)\right) \approx \bar{K}_i\left(\mathscr{C}\left(\bar{E};\bar{\vartheta}(\Gamma'),F\right)\right). \qquad \blacksquare$$

COROLLARY 9.2.8 *Assume* $E = \mathscr{C}(\mathbb{T},\mathbb{C})$.

a) If θ_1, θ_2, θ_3, $\theta_4 \in \mathbb{R}$ *such that* $\theta_1 \le \theta_2 < \theta_1 + 2\pi$*,* $\theta_3 \le \theta_4 < \theta_3 + 2\pi$ *then*

$$K_i\left(\mathscr{C}\left(E;\left\{\, e^{i\theta} \,\middle|\, \theta_1 \le \theta \le \theta_2 \right\},F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}\left(E;\left\{\, e^{i\theta} \,\middle|\, \theta_3 \le \theta \le \theta_4 \right\},F\right)\right)$$

for every $i \in \{0,1\}$.

b) Let $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 \le \theta_2 < \theta_1 + 2\pi$ and let Γ be a closed set of

$$\mathbb{T} \setminus \left\{ e^{i\theta} \;\middle|\; \theta_2 < \theta < \theta_1 + 2\pi \right\}$$

such that $e^{i\theta_1} \in \Gamma$ and $e^{i\theta_2} \notin \Gamma$ if $e^{i\theta_1} \neq e^{i\theta_2}$. Then

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(E;\, \Gamma, F))$$

for every $i \in \{0,1\}$. Moreover

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)) \approx \begin{cases} K_{i+1}(\mathscr{C}(E;\, \{1\}, F))^{\Gamma} & \text{if} \quad F \text{ is finite} \\ \sum\limits_{n \in \mathbb{N}} K_{i+1}(\mathscr{C}(E;\, \{1\}, F)) & \text{if} \quad F \text{ is infinite} \end{cases} .$$

c) If Γ_1, Γ_2 are closed sets of $\mathbb{T}$, not equal to $\mathbb{T}$ and such that their cardinal numbers are equal if they are finite then

$$K_i(\mathscr{C}(E; \Gamma_1, F)) \approx K_i(\mathscr{C}(E; \Gamma_2, F))$$

for all $i \in \{0,1\}$.

a) We may assume $\theta_1 \le \theta_3 < \theta_1 + 2\pi$. Put $\Omega' := [\theta_1, \sup(\theta_2, \theta_3)]$, $E' := \mathscr{C}(\Omega', \mathbb{C})$,

$$\vartheta : \Omega' \longrightarrow \mathbb{T}, \quad \alpha \longmapsto e^{i\alpha},$$

$$\phi : E \longrightarrow E', \quad x \longmapsto x \circ \vartheta .$$

Since it is possible to find an $f' \in \mathscr{F}(T, E')$ and a $(C'_n)_{n \in \mathbb{N}} \in \prod\limits_{n \in \mathbb{N}} E'_n$ with the desired properties, we get

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \;\middle|\; \theta_1 \le \theta \le \theta_2 \right\}, F\right)\right) \approx K_i\left(\mathscr{C}\left(E; \{\mathrm{e}^{\mathrm{i}\theta_3}\}, F\right)\right) .$$

by Corollary 9.2.7. Thus

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \;\middle|\; \theta_3 \le \theta \le \theta_4 \right\}, F\right)\right) \approx K_i\left(\mathscr{C}\left(E; \{\mathrm{e}^{\mathrm{i}\theta_3}\}, F\right)\right),$$

$$K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \;\middle|\; \theta_1 \le \theta \le \theta_2 \right\}, F\right)\right) \approx$$
$$\approx K_i\left(\mathscr{C}\left(E; \left\{ e^{i\theta} \;\middle|\; \theta_3 \le \theta \le \theta_4 \right\}, F\right)\right) .$$

b) If we put $\Omega' := [\theta_1, \theta_1 + 2\pi]$, $E' := \mathscr{C}(\Omega', \mathbb{C})$,

$$\vartheta : \Omega' \longrightarrow \mathbb{T}, \quad \alpha \longmapsto e^{i\alpha},$$

$$\phi : E \longrightarrow E', \quad x \longmapsto x \circ \vartheta,$$

then the first assertion follows from Corollary 9.2.7. If Γ is finite then the last assertion follows now from a) (and Corollary 6.2.10 b) and Proposition 7.3.1 b)).

Assume now Γ infinite. Then $\Omega_0 := \mathbb{T} \setminus \Gamma$ is the union of a countable set of open intervals. Let Ξ be the set of finite such intervals ordered by inclusion and for every $\Theta \in \Xi$ let Ω_Θ be the union of the intervals of Θ and $\Gamma_\Theta := \mathbb{T} \setminus \Omega_\Theta$. By the above,

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma_\Theta, F)) \approx K_{i+1}(\mathscr{C}(E; \{1\}, F))^\Theta$$

for every $\Theta \in \Xi$. We get an inductive system of E-modules with $\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)$ as inductive limit. By Theorem 6.2.12 and Theorem 7.3.6, $K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F))$ is the inductive limit of $K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma_\Theta, F))$ for Θ running through Ξ, which proves the assertion.

c) follows from b). ■

Remark. Let δ_0 and δ_1 be the group homomorphisms from the six-term sequence associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F) \longrightarrow \mathscr{C}(E; \mathbb{T}, F) \longrightarrow \mathscr{C}(E; \Gamma, F) \longrightarrow 0.$$

Then δ_0 and δ_1 do not coincide with the group isomorphism

$$K_i(\mathscr{C}_0(E; \mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(E; \Gamma, F))$$

from Corollary 9.2.8 b).

COROLLARY 9.2.9 *If Ω is a compact space such that $E = \mathscr{C}(\Omega \times \mathbb{T}, \mathbb{C})$ then*

$$K_i(\mathscr{C}_0(E; \Omega \times (\mathbb{T} \setminus \{1\}), F)) \approx K_{i+1}(\mathscr{C}(E; \Omega \times \{1\}, F))$$

for every $i \in \{0, 1\}$. ■

COROLLARY 9.2.10 *If the spectrum of E is $\mathbb{B}_n$ for some $n \in \mathbb{N}$ then $K_i(\mathscr{C}_0(E; \mathbb{B}_n \setminus \{0\}, F)) = \{0\}$ and*

$$K_i(\mathscr{C}_0(E; \{\, \alpha \in \mathbb{R}^n \mid 0 < \|\alpha\| < 1 \,\}, F)) \approx K_{i+1}(\mathscr{C}(E; \mathbf{S}_{n-1}, F))$$

for every $i \in \{0, 1\}$. ■

COROLLARY 9.2.11 *Let $(k_j)_{j\in J}$ be a finite family in* $\mathbb{N}$, *Ω' the topological sum of the family of balls $(\mathbb{B}_{k_j})_{j\in J}$, and Ω the compact space obtained from Ω' by identifying the centers of theses balls. If ω denotes the point of Ω obtained by this identification and S denotes the union of $(\mathbb{S}_{k_j-1})_{j\in J}$ in Ω and if $E = \mathscr{C}(\Omega,\mathbb{C})$ then*

$$K_i(\mathscr{C}_0(E;\Omega\setminus\{\omega\},F)) = \{0\},$$

$$K_i(\mathscr{C}_0(E;(\Omega\setminus(\{\omega\}\cup S),F)) \approx K_{i+1}(\mathscr{C}(E;S,F))$$

for every $i\in\{0,1\}$.

If we denote by $\vartheta : \Omega' \longrightarrow \Omega$ the quotient map, by Γ the subset of Ω' formed by the centers of the balls $(\mathbb{B}_{k_j})_{j\in J}$, and by Γ' the union of $(\mathbb{S}_{k_j-1})_{j\in J}$ ($\Gamma'\subset\Omega'$) then the assertions follow from Proposition 9.2.6 b), c) and Corollary 9.2.7. ∎

LEMMA 9.2.12 *Let S be a finite group, $g\in\mathscr{F}(S,E)$, and Ω the spectrum of E.*

a) If there is an $\omega_0\in\Omega$ and a family $(\theta(s,t))_{s,t\in S}$ of selfadjoint elements of E such that

$$\theta(r,s)+\theta(rs,t) = \theta(r,st)+\theta(s,t), \quad g(s,t) = e^{i\theta(s,t)}(g(s,t)(\omega_0))$$

for all $r,s,t\in S$ then there is a $\lambda\in\Lambda(S,\mathbb{C})$ with $(g\delta\lambda)(s,t) = g(s,t)(\omega_0)$ for all $s,t\in S$.

b) If Ω is totally disconnected then there is a $\lambda\in\Lambda(S,E)$ such that

$$((g\delta\lambda)(s,t))(\Omega)$$

is finite for all $s,t\in S$.

a) For every $u\in[0,1]$ put

$$g_u : S\times S \longrightarrow Un\,E, \quad (s,t)\longmapsto e^{iu\theta(s,t)}(g(s,t)(\omega_0)).$$

Then

$$[0,1]\longrightarrow\mathscr{F}(S,E), \quad u\longmapsto g_u$$

is a continuous map with $g_1 = g$ and $g_0(s,t) = g(s,t)(\omega_0)$ for all $s,t \in S$. By Lemma 9.2.4 a),b), there are

$$0 = u_0 < u_1 < \cdots < u_{k-1} < u_k = 1$$

and a family $(\lambda_j)_{j\in\mathbb{N}_k}$ in $\Lambda(S,\mathbb{C})$ such that $g_{u_{j-1}} = g_{u_j}\delta\lambda_j$ for every $j \in \mathbb{N}_k$. We prove by induction that

$$g_{u_{l-1}} = g\prod_{j=l}^{k}\delta\lambda_j$$

for all $l \in \mathbb{N}_k$. This is obvious for $l = k$. Assume the identity holds for $l \in \mathbb{N}_k$, $l > 1$. Then

$$g\prod_{j=l-1}^{k}\delta\lambda_j = \left(g\prod_{j=l}^{k}\delta\lambda_j\right)\delta\lambda_{l-1} = g_{u_{l-1}}\delta\lambda_{l-1} = g_{u_{l-2}}\,,$$

which finishes the proof by induction. If we put

$$\lambda := \prod_{j=1}^{k}\lambda_j \in \Lambda(S,\mathbb{C})$$

then by the above

$$g\delta\lambda = g\prod_{j=1}^{k}\delta\lambda_j = g_0\,.$$

b) Let $\omega_0 \in \Omega$. Since Ω is totally disconnected and S is finite, by continuity, there is a clopen neighborhood Ω_0 of ω_0 and a family $(\theta(s,t))_{s,t\in S}$ in $Re\mathscr{C}(\Omega_0,\mathbb{C})$ such that

$$\theta(r,s)+\theta(rs,t) = \theta(r,st)+\theta(s,t)\,, \qquad g(s,t)|\Omega_0 = e^{i\theta(s,t)}(g(s,t)(\omega_0))$$

for all $r,s,t \in S$. By a), there is a $\lambda \in \Lambda(S,\mathbb{C})$ with

$$((g|\Omega_0)\delta\lambda)(s,t) = g(s,t)(\omega_0)$$

for all $s,t \in S$.

The assertion follows now from the fact that there is a finite partition $(\Omega_j)_{j\in J}$ of Ω with clopen sets such that Ω_j possesses the property of the above Ω_0 for every $j \in J$. ■

PROPOSITION 9.2.13 *If the spectrum of E is totally disconnected then there is a $\lambda \in \Lambda(T,E)$ such that $((f\delta\lambda)(s,t))(\Omega)$ is finite for all $s,t \in T$.*

By Lemma 9.2.12 b), for every $m \in \mathbb{N}$ there is a $\lambda_m \in \Lambda(S_m, E)$ such that $((g_m \delta \lambda_m)(s,t))(\Omega)$ is finite for all $s,t \in S_m$. If we put

$$\lambda : T \longrightarrow Un\, E\,, \quad t \longmapsto \lambda_m(t) \quad \text{if} \quad t \in S_m$$

then λ has the desired properties. ■

PROPOSITION 9.2.14 *Assume that T, f, and $(C_n)_{n\in\mathbb{N}}$ satisfy the conditions of* Example 5.0.4 *and of its* Remark 1 *and that the spectrum Ω of E is simply connected.*

a) There is a $\lambda \in \Lambda(T,E)$ such that $(f\delta\lambda)(s,t) \in \mathbb{C}$ for all $s,t \in T$.

b) If $K_1\left(\mathscr{C}\left(\Omega, \mathbb{C}\right)\right) = \{0\}$ for the classical K_1 then $K_1\left(E\right) = \{0\}$ for the present theory.

a) follows from Lemma 9.2.12 a).

b) follows from a), Remark 1 of Example 5.0.4, and Proposition 7.1.10. ■