Chapter 2

Research Controllability and Dynamics of Movement Singularly Perturbed System

In this chapter is formulated the criterion of controllability using properties of the operator Gramm, and to deal with the evaluation of the standard deviation of the trajectory of motion of the system.

2.1 Controllability Singularly Perturbed Systems of Optimal Control with Constantly Acting External Forces

Here is investigated the properties of controllability of the system (2.1.1) with the help of operator Gram transforming infinite space in finite. Let the controlled process is described by the equation

$$\dot{y} = A(t,\mu)y + B(t,\mu)u + f(t,\mu), \tag{2.1.1}$$

$$y(t_0,\mu) = y^0 \tag{2.1.2}$$

$$y(t_1,\mu) = y^1 \tag{2.1.3}$$

where $A(t,\mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \dfrac{A_2(t)}{\mu} & \dfrac{A_3(t)}{\mu} \end{pmatrix}$, $B(t,\mu) = \begin{pmatrix} B_1(t) \\ \dfrac{B_2(t)}{\mu} \\ \end{pmatrix}$, $f(t,\mu) = \begin{pmatrix} f_1(t) \\ \dfrac{f_2(t)}{\mu} \\ \end{pmatrix}$,

$y = \begin{pmatrix} x \\ z \end{pmatrix} \in R^{n+m}$ $x \in R^n$, $z \in R^m$ - state vectors, $n \in C^k[t_0,t_1]$, ($C^k[t_0,t_1]-$ infinite space), $f_1(t) \in R^n$, $f_2(t) \in R^m$ constantly operating outside forces; $t \in [t_0,t_1]$, $\mu > 0 -$ small parameter $(0 < \mu \le 1)$.

States $x = x(t,\mu)$, $z = z(t,\mu)$ are slow and fast motion of the system (2.1.1), respectively. We assume the following assumptions regarding the parameters of the system (2.1.1):

1. Matrix $A_i(t)$ $\left(i=\overline{1,4}\right)$ - identified uniformly bounded and uniformly continuous with their derivatives.
2. All eigenvalues of the matrix $A_4(t)$ have negative real parts for all $t\in[t_0,t_1]$.

For linear systems usually criterion controllability are formulated using the properties of a linear operator [4].

First, consider the simplest case when the matrix $A(t,\mu)$ in the equation of the system (2.1.1) is equal to zero matrix. Then the dynamics of the system described by the equation

$$y=B(t,\mu)u+f(t,\mu). \tag{2.1.4}$$

We pose the problem of the choice of control $u(t)=u(t,\mu)$, which would ensure at the time satisfy the boundary conditions (2.1.3). Considering the conditions (2.1.2), (2.1.3) from the equation of motion (2.1.4) obtain the

$$y^1=y^0+\int_{t_0}^{t_1}B(t,\mu)u(s,\mu)ds+\int_{t_0}^{t_1}f(s,\mu)ds. \tag{2.1.5}$$

Then the expression $L(u)=\int_{t_0}^{t_1}B(t,\mu)u(t,\mu)dt$ can be viewed as a linear operator acting from the space $C^m[t_0t_1]$ to R^{n+m}. Because is required choose the control $u(t,\mu)$, which would satisfy the condition (2.1.5), it is easy to see, that if $y^1-y^0-\int_{t_0}^{t_1}f(s,\mu)ds$ lies in the region of the operator $L(u)$, then the desired transition to the state $y(t_1)=y^1$ available. Otherwise - is not. Therefore, to check whether state-controlled necessary to establish, whether it is in the region values of the operator $L(u)$.

Control $u(t)=u(t,\mu)$, which transfers status of the system (2.1.4) from y^0 at $t=t_0$ to y^1 exists only when the vector $y^1-y^0-\int_{t_0}^{t_1} f(s,\mu)ds$ lies in the region values of a linear transformation

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1} B(s,\mu)B'(s,\mu)ds \tag{2.1.6}$$

At one of the controls that translates system from one state into another and has the form:

$$u(t,\mu)=B'(t,\mu)\eta \tag{2.1.7}$$

where η is any solution of the equation

$$W(t_0,t_1,\mu)\eta = y^1-y^0-\int_{t_0}^{t_1} f(s,\mu)ds\,. \tag{2.1.8}$$

Now we move to a system of general form (2.1.1) when $A(t,\mu)\neq 0$. Integrating the equations of motion of the system (2.1.1) gives

$$y(t,\mu)=Y(t,t_0,\mu)y^0+\int_{t_0}^{t_1} Y(t,s,\mu)B(s,\mu)u(s,\mu)ds+\int_{t_0}^{t_1} Y(t,s,\mu)f(s,\mu)ds, \tag{2.1.9}$$

where $Y(t,s,\mu)$ - transition matrix for the equation

$$\dot{y}=A(t,\mu)y, \quad y(t_0,\mu)=y^0 \tag{2.1.10}$$

At $t=t_1$, taking into account (2.1.3) from (2.1.9) we will have equation of moment [42].

$$\alpha(\mu)=\int_{t_0}^{t_1} Y(t_0,s,\mu)B(s,\mu)u(s,\mu)ds \tag{2.1.11}$$

where $\alpha(\mu)=y^0-Y(t_0,t_1,\mu)y'-\int_{t_0}^{t_1} Y(t_0,s,\mu)f(s,\mu)ds.$

Theorem 2.1.1. For system (2.1.1) if and only if exists a control $u(t)=u(t,\mu)$, which transfers from state of the system (2.1.2) to the state (2.1.3) at $t=t_1>t_0$, when the vector $\alpha(\mu)$ belongs in the field of the values of the linear transformation

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1} Y(t_0,s,\mu)B(s,\mu)B'(s,\mu)Y(t_0,s,\mu)ds\,. \tag{2.1.12}$$

At that control

$$u(t,\mu)=-B'(s,\mu)Y(t_0,t_1,\mu)y_* \tag{2.1.13}$$

is one of the controls to ensure this transition, where the vector y_* is determined from the equation

$$W(t_0,t_1,\mu)y_*=\alpha(\mu). \tag{2.1.14}$$

Proof. We introduce the change of variable

$$\eta(t,\mu)=Y(t_0,t,\mu)y(t,\mu). \tag{2.1.15}$$

Then by the properties of the transition matrix will be

$$y(t,\mu)=Y(t,t_0,\mu)\eta(t,\mu),$$

$$\begin{aligned}&Y(t,t_0,\mu)\dot{\eta}(t,\mu)+\dot{Y}(t,t_0,\mu)\eta(t,\mu)\\&=A(t,\mu)Y(t,t_0,\mu)\eta(t,\mu)+B(t,\mu)u(t,\mu)+f(t,\mu)\end{aligned}$$

or

$$Y(t,t_0,\mu)\dot{\eta}(t,\mu)=B(t,\mu)u(t,\mu)+f(t,\mu).$$

Multiplying this equality on the left to matrix $Y(t_0,t,\mu)$ obtain the

$$\dot{\eta}(t,\mu)=Y(t_0 t,\mu)B(t,\mu)u(t,\mu)+Y(t_0,t,\mu)f(t,\mu). \qquad (2.1.16)$$

If reasoning as in the previous case, control $u(t,\mu)$ exists if and only if the set of values that can take the

$$\eta(t_1,\mu)-\eta(t_0,\mu)-\int_{t_0}^{t_1} Y(t_0,s,\mu)f(s,\mu)ds$$

belongs to the region of values of operator

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1} Y(t_0,s,\mu)B(s,\mu)B'(s,\mu)Y'(t_0,s,\mu)ds. \qquad (2.1.17)$$

Then the desired transition is possible, if we to require that there has been a

$$\eta(t_1,\mu)-\eta(t_0,\mu)-\int_{t_0}^{t_1} Y(t_0,s,\mu)f(s,\mu)ds$$
$$=Y(t_0,t_1,\mu)y^1-y^0-\int_{t_0}^{t_1} Y(t_0,s,\mu)f(t,\mu)ds=-\alpha(\mu).$$

This means that the desired transformation is possible if and only if the vector $\alpha(\mu)$ for each $\mu>0$ lies in the region values $W(t_0,t_1,\mu)$ and one of the control providing this transformation is a control (2.1.13), q.e.d.

It follows from this theorem that if $0<\mu<1$ and a t_0 for all t_1 matrix $W(t_0,t_1,\mu)$ has maximal rank, then the system (2.1.1) is completely controllable. Matrix $W\left(t_0,t_1,\mu\right)$ in shape (2.1.12) at $\mu>0$ has the following properties [65]: it is symmetric, non-negative, defined for $t_1 \geq t_0$ and satisfies:

a) matrix differential equation

$$\begin{aligned}&\dot{W}\left(t,t_1,\mu\right)=A(t,\mu)W\left(t,t,\mu\right)+W\left(t,t_1,\mu\right)A'(t,\mu)-B(t,\mu)B'(t,\mu),\\&W\left(t_1,t_1,\mu\right)=0\end{aligned} \qquad (2.1.18)$$

b) functional equation

$$W\left(t_0,t_1,\mu\right)=W\left(t_0,t,\mu\right)+Y\left(t_0,t,\mu\right)W\left(t,t_1,\mu\right)Y'\left(t_0,t,\mu\right). \qquad (2.1.19)$$

If we introduce in the form of a block matrix

$$W\left(t,t_1,\mu\right)=\begin{pmatrix} W_1(t,t_1,\mu) & W_2(t,t_1,\mu) \\ W_2^{'}(t,t_1,\mu) & \dfrac{1}{\mu}W_3(t,t_1,\mu)\end{pmatrix}, \qquad (2.1.20)$$

then the equation (2.1.18) can be rewritten as a system of three linear singularly perturbed equations are not separated variables:

$$\begin{aligned}&\dot{W}_1=A_1(t)W_1+A_2(t)W_2^{'}+W_1^{'}A_1^{'}(t)+W_2A_2(t)-B(t)B_1^{'}(t),\\&\mu\dot{W}_2=\mu A_1(t)W_2+A_2(t)W_3+W_1A_3^{'}(t)+W_2A_4^{'}(t)-B_1(t)B_2^{'}(t),\\&\mu\dot{W}_3=\mu A_3(t)W_2+A_4(t)W_3+\mu W_2^{'}A_3^{'}(t)+W_3A_4^{'}(t)+B_2(t)B_2^{'}(t),\end{aligned} \qquad (2.1.21)$$

$$W_1\left(t_1,t_1,\mu\right)=0,\ W_2\left(t_1,t_1,\mu\right)=0,\ W_1\left(t_1,t_1,\mu\right)=0 \qquad (2.1.22)$$

Theorem 2.1.2. Let matrix $H=H(t,\mu)$, $N=N(t,\mu)$ are solutions of differential equations

$$-\mu\dot{H}-\mu H\left(A_1(t)+A_2(t)H\right)+A_3(t)+A_4(t)H=0, \qquad (2.1.23)$$

$$\mu\dot{N}-\mu H\left(A_1(t)+A_2(t)H\right)N+N\left(A_4(t)-\mu HA_2(t)\right)+A_2(t)=0. \qquad (2.1.24)$$

Then the matrix

$$\tilde{W}(t,t_1,\mu)=\int_t^{t_1} G(t,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds \tag{2.1.25}$$

satisfies the matrix differential equation

$$\dot{\tilde{W}}=\tilde{A}(t,\mu)\tilde{W}+\tilde{W}\tilde{A}_1'(t,\mu)-\tilde{B}(t,\mu)\tilde{B}_1'(t,\mu), \tag{2.1.26}$$

where

$$\tilde{A}(t,\mu)=M^{-1}(t,\mu)\left(A(t,\mu)M(t,\mu)-\dot{M}(t,\mu)\right), \tag{2.1.27}$$

$$\tilde{B}(t,\mu)=M^{-1}(t,\mu)B(t,\mu),$$

$$M(t,\mu)=\begin{pmatrix} E_n & -\mu N(t,\mu) \\ H(t,\mu) & E_m-\mu H(t,\mu)N(t,\mu) \end{pmatrix},$$

$$G(t,s,\mu)=\begin{pmatrix} \Phi(t,s,\mu) & 0 \\ 0 & \Psi(t,s,\mu) \end{pmatrix},$$

$\Phi(t,s,\mu),\Psi(t,s,\mu)-$ transition matrices of homogeneous equations:

$\dot{\tilde{x}}=\left(A_1(t)+A_2(t)H(t,\mu)\right)x, \quad \mu\dot{\tilde{z}}=\left(A_4(t)-\mu H(t,\mu)A_2(t)\right)\tilde{z}$ respectively.

Proof. In the matrix equation (2.1.18) we introduce the change of variables in the form of

$$W=M(t,\mu)\tilde{W}M'(t,\mu). \tag{2.1.28}$$

Then in view of (2.1.28) from (2.1.18) we have

$$\dot{M}\tilde{W}M'+M\dot{\tilde{W}}M'+M\tilde{W}M'=AM\tilde{W}M'+M\tilde{W}M'A'+BB',$$

$$M\dot{\tilde{W}}M'=\left(AM-\dot{M}\right)\tilde{W}M'+M\tilde{W}\left(M'A'-\dot{M}'\right)-BB'.$$

Multiplying the left by the matrix M^{-1} and the right to M'^{-1} we obtain the equation (2.1.26). When the condition of the theorem matrix

$$\tilde{A}(t,\mu)=M^{-1}(t,\mu)\left(A(t,\mu)M(t,\mu)-\dot{M}(t,\mu)\right)$$

is a diagonal block matrix, i.e.

$$\tilde{A}(t,\mu)=\begin{pmatrix} A_1(t)-A_2(t)H(t,\mu) & 0 \\ 0 & \dfrac{1}{\mu}(A_4(t)-\mu H(t,\mu)A_2(t)) \end{pmatrix}.$$

We calculate the derivative of the function $\tilde{W}(t,t_1,\mu)$ by t

$$\dot{\tilde{W}}=\frac{d}{dt}\left(\int_t^{t_1}G(t,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds\right)=$$

$$-\tilde{B}(t,\mu)\tilde{B}'(t,\mu)+\tilde{A}(t,\mu)\int_t^{t_1}G(t,s,\mu)\tilde{B}(s,\mu)B'(s,\mu)G'(t,s,\mu)ds+$$

$$+\int_t^{t_1}G(t,s,\mu)B(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds\tilde{A}'(t,\mu)$$

$$=\tilde{A}(t,\mu)\tilde{W}+\tilde{W}A'(t,\mu)-\tilde{B}(t,\mu)B'(t,\mu)$$

q.e.d.

As in the previous case, if you enter the block matrix

$$\tilde{W}(t,t_1,\mu)=\begin{pmatrix} \tilde{W}_1(t,t_1,\mu) & \tilde{W}_2(t,t_1,\mu) \\ \tilde{W}_2'(t,t_1,\mu) & \dfrac{1}{\mu}\tilde{W}_3(t,t_1,\mu \end{pmatrix},$$

then the equation (2.1.26) can be rewritten as a system of singularly perturbed three equations with separated variables

$$\dot{\tilde{W}}_1=\tilde{A}_1(t,\mu)\tilde{W}_1+\tilde{W}_1\tilde{A}_1'(t,\mu)-B_1(t,\mu)\tilde{B}_1'(t,\mu),$$

$$\mu\dot{\tilde{W}}_2 = \mu\tilde{A}_1(t,\mu)\tilde{W}_2 + \tilde{W}_2'\tilde{A}_4'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_2'(t,\mu), \tag{2.1.29}$$

$$\mu\dot{\tilde{W}}_3 = \tilde{A}_4(t,\mu)\tilde{W}_3 + \tilde{W}_3\tilde{A}_4'(t,\mu) - \dot{\tilde{B}}_2(t,\mu)\tilde{B}_2'(t,\mu),$$

with the final conditions

$$\tilde{W}_1(t_1,t_1,\mu) = 0,\ \tilde{W}_2(t_1,t_1,\mu) = 0,\ \tilde{W}_3(t_1,t_1,\mu) = 0, \tag{2.1.30}$$

where $\tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu),\ \ \tilde{A}_4(t,\mu) = A_4(t) + \mu H(t,\mu)A_2(t),$

$$\tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\big(B_2(t) - \mu H(t,\mu)B_1(t)\big),$$
$$\tilde{B}_2(t,\mu) = B_2(t) + \mu H(t,\mu)B_1(t).$$

Equations included in the system (2.1.29) does not depend on each other and their solutions are matrix

$$\tilde{W}_1(t,t_1,\mu) = \int_t^{t_1} \Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_1(s,\mu)\Phi'(t,s,\mu)ds,$$

$$\tilde{W}_2(t,t_1,\mu) = \frac{1}{\mu}\int_t^{t_1} \Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds,$$

$$\tilde{W}_3(t,t_1,\mu) = \frac{1}{\mu}\int_t^{t_1} \Psi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi(t,s,\mu)ds,$$

at $\mu \to 0$ for matrix $\tilde{W}_1(t,t,\mu),\ \tilde{W}_2(t,t,\mu),\ \tilde{W}_3(t,t,\mu)$ we have the following limit relations:

$$\tilde{W}_1(t,t_1,\mu),\to\overline{W}_1(t),\ \tilde{W}_2(t,t_1,\mu),\to\overline{W}_2(t),\ \tilde{W}_3(t,t_1,\mu),\to\overline{W}_3(t),$$

uniformly in $t \in \left[t_0, t_1^*\right] \subset \left[t_0, t_1\right]$. Matrix $\overline{W}_1(t) = \int\limits_t^{t_1} \overline{\Phi}(t,s) B_0(s) B_0'(s) \overline{\Phi}(t,s) ds$ is the solution of the matrix differential equation $\dot{\overline{W}} = A_0(t)\overline{W}_1 + \overline{W}_1 A_0'(t) - B_0(t) B_0(t)$, $W(t_1, t_1) = 0$,

where $A_0(t) = A_1(t) - A_2(t) A_4^{-1}(t) A_3(t)$, $B_0(t) = B_1(t) - A_2(t) A_4^{-1}(t) B_2(t)$,

$$\overline{W}_2(t) = B_0(t) B_2'(t) A_4'^{-1}(t),$$

$$\overline{W}_3(t) = \int\limits_0^{\infty} e^{-A_4(t)\sigma} B_2(t_1) B_2'(t) e^{-A_4''(t)\sigma} d\sigma,$$

is the solution of algebraic equations

$$A_4(t) W_3(t) - \overline{W}(t)_3 A_4'(t) = B_2(t) B_2'(t).$$

2.2 The Criterion Controllability of Movement of Singularly Perturbed System

As shown in the preceding paragraph, after transformation gramiana controllability we received matrix (2.1.25). The structure of the matrix is not changed and as gramiana controllability can take the matrix (2.1.25). As in the previous case, if enter the block matrix

$$\tilde{W}(t, t_1, \mu) = \begin{pmatrix} \tilde{W}_1(t, t_1, \mu) & \tilde{W}_2(t, t_1, \mu) \\ \tilde{W}_2'(t, t_1, \mu) & \dfrac{1}{\mu} \tilde{W}_3(t, t_1, \mu) \end{pmatrix}, \qquad (2.2.1)$$

then the equation (2.1.26) can be rewritten as a system of singularly perturbed three equations with separated variables

$$\dot{\tilde{W}}_1 = \tilde{A}_1(t,\mu)\tilde{W}_1 + \tilde{W}_1 A_1'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_1'(t,\mu),$$
$$\mu\dot{\tilde{W}}_2 = \mu\tilde{A}_1(t,\mu)\tilde{W}_2 + \tilde{W}_2'\tilde{A}_4'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_2'(t,\mu) \tag{2.2.2}$$
$$\mu\dot{\tilde{W}}_3 = \tilde{A}_3(t,\mu)\tilde{W}_3 + \tilde{W}_3\tilde{A}_4(t,\mu) - \tilde{B}_2(t,\mu)\tilde{B}_2'(t,\mu),$$

$$\tilde{W}_1(t_1,t_1,\mu) = 0, \quad \tilde{W}_2(t_1,t_1,\mu) = 0, \quad \tilde{W}_3(t_1,t_1,\mu) = 0, \tag{2.2.3}$$

where $\tilde{W}_1 = \tilde{W}_1(t_1,t_1,\mu)$, $\tilde{W}_3 = \tilde{W}_3(t_1,t_1,\mu) -$ symmetric matrices sizes $n \times n$ and $m \times m$ respectively, $\tilde{W}_2 = \tilde{W}_2(t_1,t_1,\mu) -$ matrix size $n \times m$,

$$\tilde{A}_1(t,\mu) = A_1(t) + A_2(t)\mathrm{H}(t,\mu), \quad \tilde{A}_4(t,\mu) = A_4(t) + \mu\mathrm{H}(t,\mu)A_2(t),$$
$$\tilde{B}_1(t,\mu) = B_1(t) + \mathrm{N}(t,\mu)\left(B_2(t) - \mu\mathrm{H}(t,\mu)B_1(t)\right),$$
$$\tilde{B}_2(t,\mu) = B_1(t) + \mu\mathrm{H}(t,\mu)B_1(t).$$

It should be noted that under the conditions of theorem 2.1.2 the initial system (2.1.1) can be replaced by an equivalent system (1.1.25). Such a change is possible, since the matrix integral manifolds $H = H(t,\mu)$, $N = N(t,\mu)$ as the solutions of equations (1.1.20), (1.1.21) there are exists and unique (see chap. 1).

Then we can formulate the following theorem (analogous to theorem 2.1.1).

Theorem 2.2.1. For the system (1.1.25) at μ >0 if and only if there exists a control $\tilde{u}(t) = \tilde{u}(t,\mu)$, which transfers the system from the initial state $\tilde{y}(t_{0,}\mu) = \tilde{y}^0$ to the final state $\tilde{y}(t_1,\mu) = \tilde{y}^1$ (see 1.1.24) at $t = t_1 > t_0$, when the vector $\tilde{\alpha}(\mu) = M^{-1}(t_0,\mu)\alpha(\mu)$ belongs in the region of values of the linear transformation

$$\tilde{W}(t_0,t_1,\mu) = \int_{t_0}^{t_1} G(t_0,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t_0,s,\mu)ds. \tag{2.2.4}$$

At the same time the control

$$\tilde{u}(t,\mu) = -\tilde{B}'(t,\mu)G'(t_0,t,\mu)\tilde{y}_* \qquad (2.2.5)$$

is one of the controls providing this transition, where the vector is determined from the equation

$$\tilde{W}(t_0,t_1,\mu)\tilde{y}_* = \tilde{\alpha}(\mu), \qquad (2.2.6)$$

where

$$\tilde{\alpha}(\mu) = M^{-1}(t_0,\mu)\alpha(\mu) = M^{-1}(t_0,\mu)y^0 - G(t_0,t_1,\mu)M^{-1}(t_1,\mu)y^1$$
$$+\int_{t_0}^{t_1} G(t_0,s,\mu)M^{-1}(s,\mu)f(s,\mu)ds.$$

As shown by the formula (2.2.6) if the matrix $\tilde{W}(t_0,t_1,\mu)$ has maximal rank, then the control system provides translation (1.1.25) from the initial state $(t_0,\tilde{y}^0)$ to the final state $(t_1,\tilde{y}^1)$ and system (1.1.25) (and simultaneously the system (2.1.1)) is considered quite controllable. Therefore, our the nearest goal is to deduce from the system of equations (2.2.2.) the conditions that provide full controllability of the system (2.1.1.).

For sufficiently small values μ, of the equations obtained with respect $\tilde{W}_2$ and $\tilde{W}_3$ are singularly perturbed. At $\mu = 0$ we have no disturbed (degenerate) system

$$\begin{aligned} &\dot{\bar{W}}_1 = A_0(t)\bar{W}_1 + \bar{W}_1 A_0'(t) - B_0(t)B_0'(t), \quad \bar{W}_1(t_1,t_1) = 0, \\ &0 = \bar{W}_2 A_4'(t) - B_0(t)B_2'(t), \\ &0 = A_4(t_1)\bar{W}_3 + \bar{W}_3 A_4'(t) - B_2(t)B_2'(t), \end{aligned} \qquad (2.2.7)$$

where $A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t).$

The solution of the degenerate system approximates the solution of the problem (2.2.2), (2.2.3) with precision $O(\mu)$, and for $\tilde{W}_2$ and $\tilde{W}_3$ this is true outside the boundary layer [45], i.e. at away from the point $(t_1, 0)$.

Since we are interested in the values of submatrices $\bar{W}_i$ $(i=2,3)$ at the point $t=t_0$, so the value of $\bar{W}_i$ $(i=2,3)$ at the point $t=t_0$, substitute values of submatrices $\bar{W}_i$ $(i=2,3)$ at indicated the point with an accuracy $O(\mu)$.

At $t=t_0$ from (2.2.7) we have a matrix algebraic equations with constant coefficients. From the second equation can be determined immediately $W_2(t_0,t_1)$:

$$W_2(t_0,t_1) = B_0(t_0)B_2'(t_0)A_4'^{-1}(t_0) \tag{2.2.8}$$

The equation for $W_3(t_0,t_1)$ is the equation of Lyapunov:

$$A_4(t_0)\bar{W}_3(t_0,t_1) + \bar{W}_3(t_0,t_1)A_4'(t_0) = B_2(t_0)B_2'(t_0) \tag{2.2.9}$$

Since the proposal for the real parts of the eigenvalues values of matrix $A_4(t)$ negative for all $t \in [t_0, t_1]$, then the solution of the Lyapunov equation can be represented as a convergent integral [45]

$$\bar{W}_3(t_0,t_1) = \int_0^\infty e^{-A_4(t_0)\tau} B_2(t_0) B_2'(t_0) e^{-A_4'(t_0)\tau} d\tau \tag{2.2.10}$$

solutions (2.2.8) and (2.2.10) may be obtained by other ways. Let’s show it.

Decision matrix equations (2.2.2.) can be formally represented as [45]

$$\tilde{W}_1(t,t_1\mu)=\int_t^{t_1}\Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_1'(s,\mu)\Phi'(t,s,\mu)ds, \tag{2.2.11}$$

$$\tilde{W}_2(t,t_1,\mu)=\frac{1}{\mu}\int_t^{t_1}\Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds, \tag{2.2.12}$$

$$\tilde{W}_3(t,t_1,\mu)=\frac{1}{\mu}\int_t^{t_1}\Psi(t,s,\mu)\tilde{B}_2(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds. \tag{2.2.13}$$

At $\mu\to 0$ matrix $\tilde{W}_1(t_1,t_0\mu)-$ (2.2.11) tends to the solution of the first equation of the system (2.2.7), i.e.

$$\overline{W_1}(t,t_1)=\int_t^{t_1}\overline{\Phi}(t,s)B_0(s)B_0'(s)\overline{\Phi}'(t,s)ds, \tag{2.2.14}$$

where $\overline{\Phi}(t,s)-$ transition matrix for the homogeneous equation $\bar{x}(t)=A_0(t)\bar{x}(t)$,

$$A_0(t)=A_1(t)-A_2(t)A_4^{-1}(t)A_3(t),\quad B_0(t)=B_1(t)-A_2(t)A_4^{-1}(t)B_2(t).$$

We introduce a new variable $\tau=\dfrac{t-t_0}{\mu}$ to (2.2.12), (2.2.13) we note that at sufficiently small μ matrices $A_4(t_0+\tau\mu)$, $B_0(t_0+\tau\mu)$, $B_2(t_0+\tau\mu)$ are slowly varying functions in the space and they can be replaced by constant matrices $A_4(t_0), B_0(t_0), B_2(t_0)$ [45]. Then at $\mu\to 0$, $(\tau\to\infty)$ for matrix $W_i(t,t_1,\mu)$ $(i=1,2,3)$ at the point $t=t_0$ has the following limit relations:

$$\tilde{W}_1(t_0,t_1,\mu) \to \bar{W}_1(t_0,t_1) = \int_{t_0}^{t_1} \bar{\Phi}(t_0,s) B_0(s) B_0'(s) \bar{\Phi}'(t_0,s)\, ds,$$
$$\tilde{W}_2(t_0,t_1,\mu) \to \bar{W}_2(t_0,t_1) = B_0(t_0) B_2'(t_0) A_4'^{-1}(t_0), \tag{2.2.15}$$
$$\tilde{W}_3(t_0,t_1,\mu) \to \bar{W}_3(t_0,t_1) = \int_0^{\infty} e^{-A_4(t_0)\tau} B_2(t_0) B_2'(t_0) e^{-A_4'(t_0)\tau} d\tau.$$

Lemma. Let matrix $\bar{W}_1$ and $\bar{W}_3$ are nonzero, then at $\mu \to 0$ vector $\tilde{\alpha} = \tilde{\alpha}(\mu)$ will be finite value if and only if the last m components of vector $\tilde{y}_*$ at $\mu \to 0$ tends to zero.

Proof. Let the vector $\tilde{\alpha}$ limited, i.e. exist a number M, that

$$|\tilde{\alpha}_i| \le M \tag{2.2.16}$$

for all i=1,2…, $n+m$. From (2.2.6) we have the following relation

$$\tilde{W}^{-1}(t_0,t_1,\mu)\tilde{\alpha} = \tilde{y}_* \tag{2.2.17}$$

Using the formula the Frobenius [13] ratio (2.2.17) is written in the form

$$\begin{pmatrix} \omega_1 & \mu\omega_2 \\ \mu\omega_2' & \mu\omega_3 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_* \\ \tilde{z}_* \end{pmatrix}, \tag{2.2.18}$$

where $\omega_1 = P'$, $\omega_2 = -P^{-1}\tilde{W}_2 \cdot \tilde{W}_3^{-1}$, $\omega_3 = \tilde{W}_3^{-1} - \mu\tilde{W}_3^{-1}\tilde{W}\ P^{-1}\tilde{W}_2\tilde{W}_3^{-1}$,

$P = \tilde{W}_1 - \mu\tilde{W}_2\tilde{W}_3^{-1}\tilde{W}_2^1$, $\omega_i = \omega_i(t_0,t_1,\mu), i = 1,2,3;$ $\tilde{\alpha}_1, \tilde{x}_* - n -$ dimensional, $\tilde{\alpha}_2, \tilde{z}_* - m -$ dimensional vectors. From (2.2.18) we obtain the

$$\tilde{y}_*(\mu) = \begin{pmatrix} \tilde{x}_* \\ \tilde{z}_* \end{pmatrix} = \begin{pmatrix} \omega_1\tilde{\alpha}_1 + \mu\omega_2\tilde{\alpha}_2 \\ \mu(\omega'_2\,\tilde{\alpha}_1 + \omega_3\tilde{\alpha}_2) \end{pmatrix} \tag{2.2.19}$$

By the condition of the Lemma, the matrices $\bar{W}_1, \bar{W}_3 -$ are nonzero and reversible. Then for sufficiently small $\mu > 0$ matrix $\omega_i \left(i = 1,2,3\right)$ exist at $\mu \to 0$ from (2.2.19) we have

$$\tilde{y}_*\left(\mu\right) \to \bar{y}_*\left(0\right) = \begin{pmatrix} \bar{w}_1 \bar{\alpha}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{x}_*\left(0\right) \\ 0 \end{pmatrix}.$$

Prove the converse, let the last m components of the vector $\tilde{y}_*\left(\mu\right)$ at $\mu \to 0$ tends to zero. It means that m - dimensional vector $\tilde{z}_*\left(\mu\right)$ has an estimate

$$\left\|\tilde{z}_*\left(\mu\right)\right\| = O\left(\mu\right).$$

Then the vector $\tilde{Z}_*\left(\mu\right)$ can be represented as

$$\tilde{z}_*\left(\mu\right) = \mu\, \beta\left(\mu\right), \ \left\|\beta\left(\mu\right)\right\| \le M, \ \mathrm{M} - const \ . \tag{2.2.20}$$

Considering (2.2.20) from (2.2.17) we get

$$\tilde{\alpha}\left(\mu\right) = \begin{pmatrix} \tilde{W}_1 \tilde{x}_* + \mu \tilde{W}_2 \beta \\ \tilde{W}_2' \tilde{x}_* + \tilde{W}_3 \beta \end{pmatrix}.$$

Then for $\mu \to 0$,

$$\tilde{\alpha}\left(\mu\right) \to \begin{pmatrix} W_1' \bar{x}_* \\ \bar{W}_2' \bar{x}_* + W_3 \bar{\beta} \end{pmatrix}, \tag{2.2.21}$$

i.e. vector $\tilde{\alpha}\left(\mu\right)$ at $\mu \to 0$ is the ultimate value, where $\bar{x}_*, \bar{\beta} - n, \quad m -$ dimensional vectors, respectively, which do not depend on μ. The lemma is proved.

When the condition of the lemma from (2.2.6) we obtain the following equation for the submatrices $\bar{W}_1$ $\bar{W}_3$:

$$\bar{W}_1 \cdot \bar{x}_* = \bar{\alpha}_1 \tag{2.2.22}$$

$$\bar{W}_3 \cdot \bar{\beta} = \bar{\alpha}_2^*, \tag{2.2.23}$$

Where

$$\bar{\alpha}_1 = x^0 - \bar{\Phi}(t_0, t_1) x^1, \quad \bar{\alpha}_2^{\,*} = \bar{\alpha}_2 - \bar{W}_2' \cdot \bar{x}_*,$$
$$\bar{\alpha}_2 = z^0 + A_4^{-1}(t_0) A_3(t_0) x^0 ..$$

These relations can be seen at once that for sufficiently small $\mu > 0$, controllability of the two sub-systems of smaller dimension type

$$\dot{\bar{x}}(t) = A_0(t)\bar{x}(t) + B_0(t)u(t) + f_1(t) \quad \mu\dot{\tilde{z}} = A_4(t_0)\tilde{z} + B_2(t_0)u + f_2(t_0), \tag{2.2.24}$$

where $f_0(t) = f_1(t) - \mathrm{A}_2(t)\mathrm{A}_4^{-1}(t) f_2(t)$, should be controllability the complete system (2.1.1). Then from the position of the application of properties of linear operators controllability criterion for the system (2.1.1) is formulated in the following theorem.

Theorem 2.2.2. For the system (2.1.1) if and only if there exists a control $u(t) = u(t, \mu)$, which transfers the system from state (t_0, y^0) to state (t_1, y^1), when vectors $\bar{\alpha}_1 = x^0 - \Phi(t_0, t_1) x^1$, $\bar{\alpha}_2^{\,*} = \bar{\alpha}_2 - \bar{W}_2{}' \bar{x}_*$ belong to the region of values of linear transformations

$$\bar{W}_1(t_0, t_1) = \int_{t_0}^{t_1} \bar{\Phi}(t_0, s) \mathrm{B}_0(s) \mathrm{B}_0'(s) \bar{\Phi}'(t_0, s) ds, \tag{2.2.25}$$

$$\overline{W}_3\left(t_0,t_1\right)=\int_0^{\infty} e^{-A_4\left(t_0\right)\tau} B_2\left(t_0\right) B_2'\left(t_0\right) e^{-A_4'\left(t_0\right)\tau} d\tau. \qquad (2.2.26)$$

Respectively, in addition, if $\overline{x}_*^0$, $\overline{\beta}^0$ - or a solution of (2.2.22) and (2.2.23), it is possible to define control $u=u_0\left(t\right)$, which depending on the time of the partial movements mutually independent sub-systems is described by different analytic expressions and provides this transition to an accuracy $O\left(\mu\right)$, i.e. it is written in the form

$$u_0(t)=\begin{cases} \overline{u}^0(t), & t_0 \le t \le t_1 \\ \overline{u}^0(t_0)+V(\tau), & 0\le\tau\le\dfrac{t_1-t_0}{\mu}<+\infty, \end{cases} \qquad (2.2.27)$$

where $\tau=\dfrac{t-t_0}{\mu}$, $\overline{u}^0\left(t\right)=-B_0'\left(t\right)\Phi'\left(t_0,t\right)x_*^0$, $V\left(\tau\right)=-B_2'\left(t_0\right)e^{-A_4'\left(t_0\right)\tau}\beta^0$.

Note that if the vector $\overline{\alpha}_1$ belongs to the region of the linear transformation (2.2.25), the first subsystem of the system (2.2.24) is completely controllable. To prove this part of the theorem is not difficult.

Proof. The prove of the theorem hold for fast subsystem of the system (2.2.24) by means of a change of variable

$$\eta\left(t,\mu\right)=e^{A_4\left(t_0\right)\left(\frac{t_0-t}{\mu}\right)}\tilde{z}\left(t,\mu\right). \qquad (2.2.28)$$

Then

$$\tilde{z}\left(t,\mu\right)=e^{A_4\left(t_0\right)\left(\frac{t-t_0}{\mu}\right)}\eta\left(t,\mu\right) \qquad (2.2.29)$$

and $\dot{\tilde{z}}\left(t,\mu\right)=e^{A_4\left(t_0\right)\left(\frac{t-t_0}{\mu}\right)}\dot{\eta}\left(t,\mu\right)+\dfrac{1}{\mu}A_4\left(t_0\right)e^{A_4\left(t_0\right)\left(\frac{t-t_0}{\mu}\right)}\eta\left(t,\mu\right)$.

Substituting the value of $\dot{z}(t,\mu)$ to the fast subsystem (2.2.24) with (2.2.28) and (2.2.29) we obtain

$$\mu e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)}\dot{\eta}(t,\mu)=B_2(t_0)u(t)+f_2(t_0),$$

from whence

$$\mu\dot{\eta}(t,\mu)=e^{A_4(t_0)\left(\frac{t_0-t}{\mu}\right)}\left(B_2(t_0)u(t)+f_2(t_0)\right) \tag{2.2.30}$$

We introduce a new control

$$u(t)=\tilde{u}(t_0)+V(\tau), \tag{2.2.31}$$

where $\tau=\frac{t_0-t}{\mu}$. Then

$$\frac{d\eta}{d\tau}=-e^{-A_4(t_0)\tau}B_2(t_0)\bar{u}^0(t_0)-e^{-A_4(t_0)\tau}B_2(t_0)V(\tau)-e^{-A_4(t_0)\tau}f_2(t_0) \tag{2.2.32}$$

The solution of the equation can be written as

$$\begin{aligned}\eta(\tau)=\eta(0)-e^{-A_4(t_0)\tau}A_4^{-1}(t_0)\left(B(t_0)u(t_0)+f_2(t_0)\right)\\ +A_4^{-1}(t_0)\left(B(t_0)u(t_0)+f_2(t_0)\right)-\int_0^{\tau}e^{-A_4(t_0)S}B_2(t_0)V(s)ds.\end{aligned} \tag{2.2.33}$$

With the change of variables

$$\begin{aligned}&\eta^*(\tau)=\eta(\tau)+e^{-A_4(t_0)\tau}A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f(t_0)\right),\\ &\eta^*(0)=\eta(0)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)\end{aligned}$$

from (2.2.29) we obtain

$$\tilde{z}(\tau)=e^{A_4(t_0)\tau}\eta(\tau)=e^{A_4(t_0)\tau}\eta^*(\tau)-A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)$$

or $\tilde{z}(\tau)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)=e^{A_4(t_0)}\eta^*(\tau)$, from whence

$$\eta^*(\tau)=e^{-A_4(t_0)\tau}\left[\tilde{z}(\tau)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)\right]. \qquad (2.2.34)$$

From the previous lemma is well known that the control $u_0(t)$, which translates the state of the fast subsystem (2.2.24) of the z^0 at $t=t_0$ to z^1 at $t=t_1$ exists if and only if the vector $\eta^*(0)-\eta^*(\tau)$ belongs to the region of values of the matrix $\bar{W}_3(t_0,t_1)$ in (2.2.26).

To complete the desired transition, require that

$$\eta^*(0)-\eta^*(\tau_1)=-\int_0^\infty e^{-A_4(t_0)s}B(t_0)V(s)\,ds. \qquad (2.2.35)$$

Then one of the controls providing in the unmentioned transition of system has the form

$$V(\tau)=-B_2'(t_0)e^{-A_4'(t_0)\tau}\beta^*, \qquad (2.2.36)$$

where β^* is determined from the equation

$$\eta^*(0)-\eta^*(\tau_1)=\bar{W}_3(t_0,t_1)\beta^*,\ \tau_1=\frac{t_1-t_0}{\mu}.$$

Corollary 1. If the matrices (2.2.25), (2.2.26) have maximal ranks, then the system (2.1.1) is completely controllable.

Corollary 2. In the stationary case:

a) the operator $\bar{W}_1(t_1,t_0)$ (2.2.25) has full rank for any $t_1>t_0$;

b) the operator $\bar{W}_3(t_1,t_0)$ (2.2.26) has full rank if the symmetric matrix B_1B_2' is positive definite.

2.3 Estimation of the Standard Deviation of the Trajectory of the System of Movement

In this section is solved the problem of estimation of the standard deviation of motion of a singularly perturbed system. The main requirement for closed-loop system is the system to return to the zero from any state, and the value of criterion quality along any such motion should be minimized.

Consider the quadratic functional

$$J = \int_{t_0}^{t_1} y'(t)\, w(t)\, y(t)\, dt \qquad (2.3.1)$$

where $w(t) = \begin{pmatrix} w_1 & w_2 \\ w_2' & w_3 \end{pmatrix}$.

In the closed-loop optimal trajectory of system is described by homogeneous equations. Therefore avoiding complex analytical expressions and extra notation we restrict homogeneous equations, which are obtained from (2.1.1) at $u(t,\mu) = 0,\ f_1(t,\mu) = 0,\ f_2(t,\mu) = 0.$

By virtue of the equations of motion $\tilde{x}(t) = \Phi(t,t_0\mu)\tilde{x}(t_0), \quad \tilde{z}(t) = \Psi(t,t_0,\mu)\tilde{z}(t_0)$ we have

$$J = \int_{t_0}^{t} \tilde{y}'(t_0) G'(t,t_{0,}\mu) W(t) G(t,t_0\mu)\tilde{y}(t_0)\, dt = \tilde{y}'(t_0) V(t_{0,}t_{1,}\mu)\tilde{y}(t_0), \qquad (2.3.2)$$

where

$$G(t,t_{0,}\mu) = \begin{pmatrix} \Phi(t,t_{0,}\mu) & 0 \\ 0 & \Psi(t,t_{0,}\mu), \end{pmatrix},\ V(t,t_0,\mu) = \int_{t_0}^{t} G'(t,t_{0,}\mu)\tilde{W}(t) G(t,t_0\mu)\, dt, \qquad (2.3.3)$$

where $\tilde{W} = M'WM, \quad M = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix}$.

Thus, the target value J is a quadratic form $\tilde{y}(t_0)$, and $V(t,t_{0,}\mu)-$ its matrix. If there are known transition matrices $\Phi(t,t_0,\mu)$, $\Psi(t,t_0,\mu)$, then the matrix $V(t_0,t_1,\mu)$ can be calculated using the formula (2.3.3). One can show other methods of calculation. This problem can be reduced to the solution of a linear system with singular perturbations, replacing t_0 to t and differentiating expression for the $V(t,t_1,\mu)$ by t we have:

$$\begin{aligned}\frac{d}{dt}V(t,t_1,\mu) &= \frac{d}{dt}\left(\int_t^{t_1} G'(s,t,\mu)\tilde{W}(s)G(s,t,\mu)ds\right) \\ &= -\tilde{A}'(t)V(t,t,\mu)-V(t,t_1,\mu)\tilde{A}(t)-\tilde{W}(t).\end{aligned} \tag{2.3.4}$$

From the definition $V(t,t_1,\mu)$ it follows that $V(t_1,t_1,\mu)=0.$

The matrix V is divided into blocks

$$V=\begin{pmatrix} V_1 & \mu V_2 \\ \mu V_2' & \mu V_3 \end{pmatrix} \tag{2.3.5}$$

and the equation (2.3.4) in the form of the system three matrices equations:

$$\begin{aligned}\dot{V}_1 &= -\tilde{A}_1'(t)V_1 - V_1\tilde{A}_1(t)-\tilde{W}_1(t),\ V_1(t_1,t_1)=0, \\ \mu\dot{V}_2 &= -\mu\tilde{A}_1'(t)V_2 - V_2\tilde{A}_4(t)-\tilde{W}_2(t),\ V_2(t_1,t_1)=0, \\ \mu\dot{V}_3 &= -\tilde{A}_4'(t)V_3 - V_3\tilde{A}_4(t)-\tilde{W}_3(t),\ V_3(t_1,t_1)=0,\end{aligned} \tag{2.3.6}$$

which can be solved independently.

Note that the boundary conditions of differential equations are given at the not initial time but at the end of the process.

Thus, the following theorem holds.

Theorem 2.3.1. If the blocks of matrix V are the solutions of differential equations in (2.3.6), and $\tilde{x}(t)$, $\tilde{z}(t)-$ solutions of the system $\dot{\tilde{x}} = \tilde{A}_1(t)\tilde{x}, \quad \mu\dot{\tilde{z}} = \tilde{A}_4(t)\tilde{z}$ at $t_0 \le t \le t_1$, then the formula is true

$$\int_{t_0}^{t_1} \tilde{y}'(t)W(t)\tilde{y}(t)dt = \tilde{y}'(t_0)V(t_0,t_1,\mu)\tilde{y}(t_0), \tag{2.3.7}$$

where $\tilde{y}(t) = \tilde{y}(t,\mu) = \begin{pmatrix} \tilde{x}(t,\mu) \\ \tilde{z}(t,\mu) \end{pmatrix}$.

Limit task (at μ→0) for (2.3.6) has the form

$$\bar{V}_1 = -\bar{A}_1(t)\bar{V}_1 - \bar{V}_1\bar{A}_1(t) - \bar{W}_1(t), \quad \bar{V}_1(t_1,t_1) = 0, \tag{2.3.7a}$$

$$0 = -\bar{V}_2A_4(t) - \bar{W}_2(t) \tag{2.3.7b}$$

$$0 = -A_4'(t)\bar{V}_3 - \bar{V}_3A_4(t) - \bar{W}_3(t),$$

where $\bar{A}_1(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t)$,

$$\bar{W}_1(t) = W_1 + H_0'W_2' + W_2H_0 + H_0'W_3H_0,$$
$$\bar{W}_2 = W_2 + H_0'W_3, \quad \bar{W}_3 = W_3, \qquad H_0 = -A_4^{-1}A_3.$$

For small μ are possible various ways to construct an approximate solution of the system (2.3.6).

At the basic of the approximate solutions lie solutions "systems of fast movements."

$$\frac{dV_2}{d\tau} = -\mu\bar{A}_1\tilde{V}_2 - \tilde{V}_2A_4 - \bar{W}_2, \qquad \tilde{V}_2(0) = 0, \tag{2.3.8}$$

$$\frac{d\tilde{V}_3}{d\tau} = -A_4'\tilde{V}_3 - \tilde{V}_3A_4 - \bar{W}_3, \qquad \tilde{V}_3(0) = 0, \tag{2.3.9}$$

where $A_0 = A_0(t_1) \approx \tilde{A}_1(t_1 + \tau\mu)$, $\tilde{A}_4(t_1 + \tau\mu) \approx A_4(t_1) = A_4$,

$$\bar{W}_i(t_1 + \tau\mu) \approx \bar{W}_i(t_1 + \tau\mu) \approx \bar{W}_i(t_1) = \bar{W}_i \qquad i = (2,3), \ \ \tau = \frac{t - t_1}{\mu},$$

$$A_0 = A_1 - A_2 A_4^{-1} A_3.$$

Solutions (2.3.8), (2.3.9) are satisfying the zero initial conditions have the form

$$\tilde{V}_2(\tau) = \int_{\tau}^{0} e^{-\mu\bar{A}_1'(\tau-\delta)_1} \bar{W}_2 e^{-A_4\tau-\sigma} d\sigma, \tag{2.3.10}$$

$$\tilde{V}_3(\tau) = \int_{\tau}^{0} e^{-A_4'(\tau-\delta)_1} \bar{W}_3 e^{-A_4\tau-\sigma} d\sigma, \tag{2.3.11}$$

Consider the equation

$$-\mu A_1'\delta_2 - \delta_2 A_4 - \bar{W}_2 = 0, \tag{2.3.12}$$

$$-A_4'\delta_3 - \delta_3 A_4 - \bar{W}_3 = 0, \tag{2.3.13}$$

It is easy to show that if the matrix A_4 stable, then the matrices

$$\delta_2 = \int_0^{\infty} e^{\mu\bar{A}_1's} W_2 e^{A_4 s} ds, \quad \delta_3 = \int_0^{\infty} e^{A_4's} W_3 e^{A_4 s} ds$$

are the unique solutions of the equations (2.3.12) and (2.3.13) respectively. If the solution (2.3.10), (2.3.11) at $\tau \to -\infty$ tend to solutions of the equations (2.3.12), (2.3.13), then the well-known theorem Tikhonov, we can say that the initial value $(\tilde{V}_2(0), \tilde{V}_3(0)) = (0,0)$ belongs to the region of influence of the rest point (δ_2 , δ_3).

This raises the question: which kind of conditions the functions $\tilde{V}_2, \tilde{V}_3$ at $\tau \to -\infty$ tend to solutions of the equations (2.3.12), (2.3.13)?

On this task a positive response given by the following theorem.

Theorem is given for (2.3.11) and (2.3.13).

Theorem 2.3.2. Let A_4 - stable matrix. Then $\tilde{V}_3(\tau) \to \delta_3$ at $\tau \to -\infty$, if and only if the equality

$$\int_{\tau}^{0} e^{A_4'\sigma} \bar{W}_3 e^{A_4\sigma} d\sigma = e^{A_4'\tau} \delta_3 e^{A_4\tau} - \delta_3, \tag{2.3.14}$$

where δ_3 - solution of the equation (2.3.13).

Proof. Let the equality (2.3.14) function $\tilde{V}_3$ is written in the form

$$\tilde{V}_3(\tau) = e^{-A_4'\tau} \int_{\tau}^{0} e^{A_4'\sigma} \bar{W}_3 e^{A_4'\sigma} d\sigma e^{-A_4\tau} \tag{2.3.15}$$

Considering (2.3.14) from (2.3.15) we obtain

$$e^{-A_4'\tau} \int_{\tau}^{0} e^{A_4'\tau} \bar{W}_3 e^{A_4\tau} \cdot d\sigma \cdot e^{-A_4\tau} = \delta_3 - e^{-A_4'\tau} \delta_3 e^{-A_4\tau} \tau.. \tag{2.3.16}$$

Since by hypothesis of theorem the matrix A_4 is stable and from this follows that for $\tau \to -\infty$ $\tilde{V}_3(\tau) \to \delta_3$.

Suppose now, on the contrary: $\tilde{V}_3(\tau) \to \delta_3$ at $\tau \to -\infty$, where δ_3 – solution of equation (2.3.13). If so, then the integral (2.3.14) can be represented in the form

$$e^{-A_4'\tau}\int_{\tau}^{0} e^{A_4'\tau}\overline{W}_3 e^{A_4\tau}\cdot d\sigma\cdot e^{-A_4\tau}=\delta_3-e^{-A_4'\tau}\delta_3 e^{-A_4\tau}\tau. \qquad (2.3.16)$$

From this follows the equation (2.3.14). Now we show the validity of the equality (2.3.14) that δ_3 - solution of the equation (2.3.13).

Differentiating both sides of (2.3.14) by τ we have:

$$-e^{A_4'\tau}\overline{W}_3 e^{A_4\tau}=A_4' e^{A_4\tau}\delta_3 e^{A_4\tau}+e^{-A_4'\tau}\delta_3 e^{A_4\tau}A_4.$$

Multiplying this equality on the left by the matrix $e^{-A'\tau}$, right to matrix $e^{-A_4\tau}$, obtain the equivalent equation:

$$-\overline{W}_3=e^{-A_4'\tau}A_4 e^{A_4'\tau}\delta_3+\delta_3 e^{A_4\tau}A_4 e^{-A_4\tau}.$$

Considering the property of the matrix exponential for constant matrix A_4: $e^{A_4\tau}A_4=A_4 e^{A_4\tau}$, we have from the last

$$-A_4'\delta_3-\delta_3 A_4-\overline{W}_3=0.$$

By assumption δ_3 is a solution of (2.3.13), and therefore is obtained the identity.

The above theorem is valid for (2.3.10) (2.3.12).

Thus, the estimate of the integral reduces to the solution of algebraic equations (2.3.12), (2.3.13) in the semi-infinite interval $(0,\infty)$. Following Tikhonov's theorem, we arrive at the following conclusion:

If a) the matrices $A_i(t)$ $(i=\overline{1,4})$ uniformly bounded and uniformly continuous together with its derivatives at $t\in[t_0,t_1]$;

b) $A_4(t)$ - stable matrix at $t \in [t_0, t_1]$, then exists a number μ_0 such that when $0 < \mu < \mu_0$ the solution of (2.3.14) exists and is unique in the segment $t_0 \le t \le t_1$.

The solution of problems (2.3.7a), (2.3.8), (2.3.9) can serve as the asymptotic behavior of solutions of (2.3.6) and when assessing the value of the integral (2.3.1) provide more accurate results than solutions problems (2.3.7).