Chapter 3

Non-Static Spherically Symmetry

In this chapter the theory of gravitation in flat space-time is applied to non-static, spherically symmetric bodies. The results of this chapter are contained in the article [Pet 92b].

3.1 The Field Equations

The line-element is given by the metric (2.2). The proper time can be written

$$
\begin{aligned}
(cd\tau)^2 &= -A(r,t)(dr)^2 - B(r,t)r^2((d\vartheta)^2 + \sin^2\theta(d\varphi)^2) \\
&\quad + C(r,t)(dct)^2 - 2D(r,t)drdct
\end{aligned}
\tag{3.1}
$$

By a transformation of time

$$ct = F(r, c\tilde{t}) \tag{3.2}$$

we can eliminate the expression with $drdct$.With the notation $(\tilde{x}^i) = (r, \vartheta, \varphi, c\tilde{t})$ the line-element can now be written in the form

$$(ds)^2 = -\eta_{ij} d\tilde{x}^i d\tilde{x}^j \tag{3.3a}$$

where

$$
\begin{aligned}
&\eta_{11} = 1 - (\frac{\partial F}{\partial r})^2, \eta_{22} = r^2, \eta_{33} = r^2 \sin^2\theta, \eta_{44} = (\frac{\partial F}{\partial c\tilde{t}})^2 \\
&\eta_{14} = \eta_{41} = \frac{\partial F}{\partial r}\frac{\partial F}{\partial c\tilde{t}}, \eta_{ij} = 0(else)
\end{aligned}
\tag{3.3b}
$$

The proper time is now given by

$$(cd\tau)^2 = -g_{ij} d\tilde{x}^i d\tilde{x}^j \tag{3.4a}$$

where

$$
\begin{aligned}
&g_{11} = 1/f(r, c\tilde{t}), g_{22} = r^2/g(r, c\tilde{t}) \\
&g_{33} = r^2 \sin^2\vartheta/g(r, c\tilde{t}), \\
&g_{44} = -1/h(r, c\tilde{t}), g_{ij} = 0(i \neq j)
\end{aligned}
\tag{3.4b}
$$

with new functions $f(r, c\tilde{t})$, $g(r, c\tilde{t})$ and $h(r, c\tilde{t})$. The energy-momentum tensor of matter described by perfect fluid (1.28) where ρ, p and (u^i) are functions of r and $\tilde{t}$. We have by virtue of spherical symmetry for a collapsing body

$$u^2 = u^3 = 0. \tag{3.5}$$

It follows with $u^i = \frac{d\tilde{x}^i}{d\tau}$ and

$$-g_{ij}u^i u^j = c^2$$

by virtue of (3.5) and (3.4b)

$$u^4 = c\left[h\left(\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)\right]^{1/2}. \tag{3.6}$$

Hence, we consider a non-static spherically symmetric body with only a radial velocity. The energy-momentum tensor of matter (1.28) has by virtue of (3.4), (3.5) and (3.6) the form

$$\begin{aligned}
T(M)_j^i &= (\rho + p)\frac{1}{f}(u^1)^2 + pc^2, (i = j = 1)\\
&= \quad pc^2, (i = j = 2{,}3)\\
&= -(\rho + p)c^2\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right) + pc^2, (i = j = 4)\\
&= -(\rho + p)cu^1\left(\frac{1}{h}\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)^{1/2}, (i = 1, j = 4)\\
&= (\rho + p)cu^1\frac{1}{f}\left(h\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)^{1/2}, (i = 4, j = 1)\\
&= 0, \quad (i \neq j)
\end{aligned} \tag{3.7}$$

Put for any function $\beta(r, \tilde{t})$ define

$$\beta_{(1)} = \partial\beta/\partial r, \beta_{(4)} = \partial\,\beta/\partial(c\tilde{t}), \beta_{(14)} = \beta_{(41)} = \partial^2\beta/\partial r\partial\,(c\tilde{t}).$$

For $i = 1, 4$ put

$$L_{ij} = \frac{f_{(i)}}{f}\frac{f_{(j)}}{f} + 2\frac{g_{(i)}}{g}\frac{g_{(j)}}{g} + \frac{\left(F_{(4)}^2 h\right)_{(i)}}{F_{(4)}^2 h}\frac{\left(F_{(4)}^2 h\right)_{(j)}}{F_{(4)}^2 h} - \frac{1}{2}\left(\frac{f_{(i)}}{f} + 2\frac{g_{(i)}}{g} + \frac{\left(F_{(4)}^2 h\right)_{(i)}}{F_{(4)}^2 h}\right)\left(\frac{f_{(j)}}{f} + 2\frac{g_{(j)}}{g} + \frac{\left(F_{(4)}^2 h\right)_{(j)}}{F_{(4)}^2 h}\right) \tag{3.8}$$

The energy-momentum tensor (1.35) of the gravitational field can be written by virtue of (3.4b) and (3.3b)

$$T(G)^i_j = \frac{1}{8\kappa F_{(4)} g (fh)^{1/2}}$$

$$\times\left\{\frac{f}{2}L_{11} + \frac{h}{2}L_{44} - \frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2 - f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2 - \frac{2}{r^2}g\left(\frac{(f-g)^2}{fg} - \frac{g}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2\right)\right\} \quad \text{(i=j=1)}$$

$$\times\left\{-\frac{f}{2}L_{11} + \frac{h}{2}L_{44} + \frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2 - f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2\right\} \quad (\mathrm{i} = \mathrm{j} = 2, 3)$$

$$\times\left\{-\frac{f}{2}L_{11} - \frac{h}{2}L_{44} + \frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2 + f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2 - \frac{2}{r^2}g\left(\frac{(f-g)^2}{fg} - \frac{g}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2\right)\right\} \quad \text{(i=j=4)} \tag{3.9}$$

$$\times\left\{fL_{14} - 2\frac{f^2}{h}\frac{F_{(11)}}{F_{(4)}}\frac{F_{(14)}}{F_{(4)}}\right\} \quad \text{(i=1, j=4)}$$

$$\times\left\{-hL_{14} + 2f\frac{F_{(11)}}{F_{(4)}}\frac{F_{(14)}}{F_{(4)}}\right\} \quad \text{(i=4, j=1)}$$

$$\times\{0\} \quad \text{(else)}$$

We get from (1.21b) with (1.10)

$$T(\Lambda)^i_j = -\frac{\Lambda}{2\kappa}\frac{1}{F_{(4)} g (fh)^{1/2}}\delta^i_j. \tag{3.10}$$

Let us define the following differential operators L_1 and L_2 of order two:

$$L_1(y) := \frac{1}{r^2 F_{(4)}}\left\{\frac{\partial}{\partial r}\left(r^2\frac{f}{g(fh)^{1/2}}\frac{y_{(1)}}{y}\right) - \frac{\partial}{\partial c\tilde{t}}\left(r^2\frac{h}{g(fh)^{1/2}}\frac{y_{(4)}}{y}\right)\right\} \tag{3.11a}$$

$$L_2(y) := \frac{1}{r^2 F_{(4)}}\left\{\frac{\partial}{\partial r}\left(r^2\frac{f^2}{F_{(4)} h g (fh)^{1/2}}y_{(1)}\right) - \frac{\partial}{\partial c\tilde{t}}\left(r^2\frac{f}{F_{(4)} g (fh)^{1/2}}y_{(4)}\right)\right\} \tag{3.11b}$$

Then, the field equations (1.24) with (1.23a) and (1.23c) have by virtue of (3.4b), (3.7), (3.9) and (3.10) the following form

$$L_1(f) = \frac{1}{F_{(4)}g(fh)^{1/2}}\left\{\frac{2}{r^2}\frac{f^2-g^2}{f} + f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2 - 2\frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2 + \frac{1}{2}fL_{11} + 2\Lambda\right\} \\ +2\kappa\left[(\rho-p)c^2 + 2(\rho+p)\frac{1}{f}(u^1)^2\right] \tag{3.12a}$$

$$L_1(g) = \frac{1}{F_{(4)}g(fh)^{1/2}}\left\{-\frac{2}{r^2}\frac{g(f-g)}{f} - \frac{2}{r^2}\frac{g^2}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2 + 2\Lambda\right\} +2\kappa(\rho-p)c^2 \tag{3.12b}$$

$$L_1\left(F_{(4)}^2 h\right) = \frac{1}{F_{(4)}g(fh)^{1/2}}\left\{\frac{2}{r^2}\frac{g^2}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2 + \frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2 - \frac{1}{2}hL_{44} + 2\Lambda\right\} \\ -2\kappa\left[(\rho+3p)c^2 + 2(\varrho+p)\frac{1}{f}(u^1)^2\right] \tag{3.12c}$$

$$L_2\left(F_{(1)}\right) = \frac{1}{F_{(4)}g(fh)^{1/2}}\left\{\frac{2}{r^2}\frac{g^2}{h}\frac{F_{(1)}}{F_{(4)}} + f\frac{F_{(14)}}{F_{(4)}}\left(2\frac{f}{h}\frac{F_{(11)}}{F_{(4)}} - \frac{F_{(44)}}{F_{(4)}}\right) - \frac{1}{2}fL_{14}\right\} \\ +4\kappa(\rho+p)cu^1\left(\frac{1}{h}\left(1+\frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)^{1/2}. \tag{3.12d}$$

The field equations (3.12) are four partial differential equations for the four unknown functions f, g, h and F defining by the use of (3.4b) the gravitational potentials $\left(g_{ij}\right)$.

3.2 Equations of Motion and Energy-Momentum Conservation

In flat space-time theory of gravitation we have in addition to the field equations the equations of motion (1.29a) and the conservation law (1.25a) of the whole energy-momentum. One of these equations follows by the other one and can be omitted. The equations of motion (1.29a) yield by the use of (3.3b), (3.4b) and (3.7) the two equations

$$\begin{aligned}&\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T(M)_1^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T(M)_1^4\right)\right\}\\&=-\frac{1}{2}\frac{f_{(1)}}{f}T(M)_1^1+\left(\frac{2}{r}-\frac{g_{(1)}}{g}\right)T(M)_2^2-\frac{h_{(1)}}{h}T(M)_4^4\end{aligned} \tag{3.13a}$$

$$\begin{aligned}&\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T(M)_4^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T(M)_4^4\right)\right\}\\&=-\frac{1}{2}\frac{f_{(4)}}{f}T(M)_1^1-\frac{g_{(4)}}{g}T(M)_2^2-\frac{1}{2}\frac{h_{(4)}}{h}T(M)_4^4\end{aligned} \tag{3.13b}$$

Equation (1.25) implies the following two equations for the whole energy-momentum

$$\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T_1^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T_1^4\right)\right\}-\frac{2}{r}T_2^2-\frac{F_{(11)}}{F_{(4)}}T_4^1-\frac{F_{(14)}}{F_{(4)}}T_4^4 \tag{3.14a}$$

$$\frac{\partial}{\partial r}(r^2T_4^1)+\frac{\partial}{\partial c\tilde{t}}(r^2T_4^4)=0. \tag{3.14b}$$

The equations (3.12), (3.13) and (3.14) describe a spherically symmetric collapsing body where one of the equations (3.13) or (3.14) can be omitted. Furthermore, $\Lambda = 0$ is assumed for a star by virtue of its smallness. In general one replaces ρ by $\rho + \rho\Pi$ where Π denotes the specific internal energy and one adds the conservation law of matter (1.29b) and an equation of state of the form

$$p = p(\rho, \Pi). \tag{3.15}$$

Hence, we have eight unknown functions f, g, h, ρ, p Π and u^1 (u^4 follows by (3.6)) depending on r and $\tilde{t}$ and eight independent equations (3.12) (four equations), (3.13) or (3.14) (two equations), (1.29b) (one equation), and (3.15) (one equation).

A solution of these equations for a collapsing star is at present time not known, also numerical solutions are not computed. Therefore, it is an open question whether black holes exist or not by the use of this flat space-time theory of gravitation.

The corresponding equations by Einstein's general theory of relativity are stated e.g. in the papers [*May* 66] and [*Mis* 64]. They are simpler than the above ones because Einstein's theory allows to reduce the number of unknown functions by suitable transformations.